Math 331 Homework: Day 16

Practice and Reading

1. The next big idea: uniform continuity. Read Section 2.6 carefully. Least upper bounds turn up again in the intermediate value and max-min theorems. **Do the Preparation on the back of the sheet.**

2. Practice: Page 74 and 75 #2.4.2, 3, 6, and 8 and Page 82 #2.5.1. Give a proof of Theorem 2.4.3.

Hand In Wednesday

Most of these are short and simply involve the various definitions of limit or the definition of continuity.

1. Prove (using the definition of one-sided limit) that $\lim_{x\to 4^+} \sqrt{x^2-16} = 0$.

2. Read all parts. Similar results hold for products and quotients. Also similar results hold for $x \to a^-$ or $x \to -\infty$. Adapt the proofs of our earlier limit properties to the new types of limits.

For parts (a) and (b): Suppose that $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to+\infty} g(x) = M$.

a) [WS] Prove that $\lim_{x\to+\infty}\big[f(x)+g(x)\big] = L+M$.

b) [H] If $c \neq 0$ is a constant, show that $\lim_{x\to+\infty} cf(x) = cL$. (The case $c = 0$ is obviously true, right?)

For parts (c) and (d): Suppose that $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^+} g(x) = M$.

c) [H] Prove that $\lim_{x\to a^+}\big[f(x)+g(x)\big] = L+M$.

d) [WS] If $c \neq 0$ is a constant, show that $\lim_{x\to a^+} cf(x) = cL$. (The case $c = 0$ is obviously true, right?)

3. Prove that $f(x) = -2x^2 - 1 - \sqrt{x}$ is continuous on $[0,\infty)$. Split the proof into two cases: $a > 0$ and the case $a = 0$ where you need to show that $\lim_{x\to 0^+} f(x) = f(0)$. Use the previous problem, Theorem 2.4.3, Examples 2.2.1 and 2.4.1, and other results to justify your steps. Avoid ϵ and δ.

4. a) Prove: If $m \in \mathbb{N}$, then $\lim_{x\to+\infty} \frac{1}{x^m} = 0$. (Use ϵ. You can do this directly without using induction).

b) Show that if p is the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\displaystyle\lim_{x\to+\infty} \frac{p(x)}{x^n} = a_n$. Hint: No ϵ proof is needed. Do the division term-by-term and then use part (a) and Problem 2 above.

c) Show that the conclusion in (b) can be rewritten as: For any $\epsilon > 0$, there is an N such that if $x > N$, then $a_n - \epsilon < p(x)/x^n < a_n + \epsilon$.

d) Prove: If $a_n > 0$, there is an N' such that if $x > N'$, then $\frac{a_n}{2} < p(x)$. This result says that for sufficiently large x, the values of a polynomial $p(x)$ are bounded away from 0. (Hint: In (c) take $\epsilon = \frac{a_n}{2}$ and let $N' = \max\{N, 1\}$.) [This result is used for Corollary 2.6.2 on page 83 where $a_n = 1$.]

5. This problem explores composition and limits. You may wish to reread the proof of Theorem 2.5.3, but this is far easier. This particular result is needed to prove l'Hôpital's Rule.

a) Prove: If c is a function such that $a < c(x) < x$ for all $x > a$, then $\lim_{x\to a^+} c(x) = a$.

b) Prove: If $\lim_{x\to a^+} f(x) = L$ and if $y = c(x)$ is a function such that $a < c(x) < x$ for all $x > a$, then $\lim_{x\to a^+} f(c(x)) = L$. Hint: If you are having a hard time "seeing" this, substitute y for $c(x)$.

6. a) Define the following limit expression: $\displaystyle\lim_{x\to a} f(x) = +\infty$

b) Use your definition to show that $\displaystyle\lim_{x\to 0} \frac{2}{|x|} = +\infty$.

7. Prove: If $f(x)$ is continuous at $x = a$, then $|f(x)|$ is continuous at $x = a$. No ϵ. Use previous work!

8. [WS]: Prove Theorem 2.5.2 (a). [H]: Prove Theorem 2.5.2 (b). See the proof of (c) in the text.

9. Suppose that $f(x)$ and $g(x)$ are continuous on $[a,b]$ and that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there is a point $x \in [a,b]$ with $f(x) = g(x)$.

10. a) Suppose that f is a function such that for any $x, y \in \mathbb{R}$ we have $f(x+y) = f(x) + f(y)$. Prove that $f(0)$=0. (Hint: $0 + 0 = 0$.)

b) With f as in part (a) prove: If f is continuous at 0 then it is continuous at any point a. Hint: $x = (x-a)+a$. OR Consider an alternative definitions of continuity.

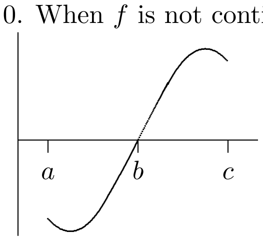
Self-Study

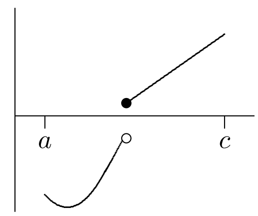
1. Carefully define the expression: $\lim_{x \to +\infty} f(x) = +\infty$. Use your definition to show that $\lim_{x \to +\infty} x^2 = +\infty$.

2. Suppose that f is defined everywhere and bounded (i.e. $|f(x)| < M$ for some fixed M). Show that the function $g(x) = xf(x)$ is continuous at $x = 0$. [How can you use this in problem 6 of §2.6?]

Class Prep: The Intermediate Value Theorem. I need a volunteer to present this next time.

On an intuitive level, the graph of a continuous function f is an "unbroken" curve. If f is continuous on the closed interval $[a, c]$ with $f(a) < 0$ and $f(c) > 0$, geometric intuition tells us that there is at least one point b between a and c such that $f(b) = 0$. When f is not continuous, clearly such a point b need not exist.





The Intermediate Value Theorem. Assume that f is continuous on the closed interval $[a, c]$ and let y be any real number such that

$$f(a) < y < f(c).$$

Then there exists at least one number b in the interval $[a, c]$ such that $f(b) = y$.

Proof: Here's the first part of the proof. Justify each of the steps. It should remind you of the proof of the Bolzano-Weierstrass Theorem. The proof makes use of the LUB Axiom and continuity, a very powerful combination. Let

$$S = \{x \in [a, c] \mid f(x) < y\}.$$

1. S is nonempty because ____.

2. S has an upper bound because ____.

3. S has a least upper bound b because ____.

4. $b \geq a$ because ____________________ and $b \leq c$ because ____________________.

5. So $b \in [a, c]$ and therefore $f(b)$ is defined. By Axiom __________ there are three possibilities: (i) $f(b) - y < 0$; (ii) $f(b) - y > 0$; or (iii) $f(b) - y = 0$. First we show that (i) is impossible.

6. a) By contradiction. Assume (i) holds: $f(b) < y$. Observe that $b \neq c$ because ____________________.
b) So $b < c$ because ____.

7. Let $\epsilon = y - f(b)$. Then $\epsilon > 0$ because ____.

8. There is a $\delta > 0 \ni$ if $|x - b| < \delta$, then $|f(x) - f(b)| < \epsilon$ because ____.

9. Pick x satisfying $b < x < b + \delta$. (Note: Here's where $b < c$ is used: $b + \delta < c$.) For any such x,

$$f(x) - f(b) \leq |f(x) - f(b)| < \epsilon.$$

So $f(x) < y$ because ____. (Hint: What's ϵ?)

10. $x \in S$ because ____.

11. Contradiction because (see #9) ____. So (i) is false.
I need another volunteer to modify the proof above to showing (ii) in part 5 is false. That would leave (iii) which is what we want to prove.