# Math 331 Homework: Day 17

## Practice and Reading

**1.** **a)** New: Page 88, #2, 5(a), 7(a).

**b)** Review: Page 74 and 75 #2.4.2, 3, 6, and 8 and Page 82 #2.5.1. Give a proof of Theorem 2.4.3.

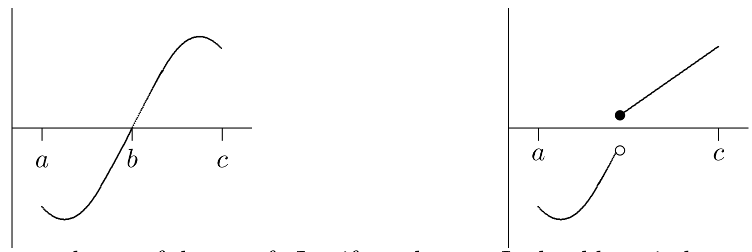
**2.** [Extra Credit.] A **fixed point** for a function is a point where $f(x) = x$. Suppose that $f : [0, 1] \to [0, 1]$ and $f$ is continuous. (This notation means that $[0, 1]$ is both the domain and range for $f$, i.e., $0 \le f(x) \le 1$ for all $x \in [0, 1]$.) Prove that it must have a fixed point in $[0, 1]$. (Hint: use the previous problem.)

## The Intermediate Value Theorem: Part Two of Proof.

Assume that $f$ is continuous on the closed interval $[a, c]$ and let $y$ be any real number such that

$$f(a) < y < f(c).$$

Then there exists at least one number $b$ in the interval $[a, c]$ so that $f(b) = y$.




*Proof*: Here's the second part of the proof. Justify each step. It should remind you of the proof of the H-B and B-W Theorems. The proof makes use of the LUB Axiom and continuity, a very powerful combination. Let

$$S = \{x \in [a, c] \mid f(x) < y\}.$$

**1.** $S$ is non-empty because ________________________________.

**2.** $S$ has an upper bound because ________________________________.

**3.** $S$ has a least upper bound $b$ because ________________________________.

**4.** $b \ge a$ because ________________ and $b \le c$ because ________________.

**5.** So $b \in [a, c]$ and therefore $f(b)$ is defined. By Axiom __________ there are three possibilities: (i) $f(b) < y$; (ii) $f(b) > y$; or (iii) $f(b) = y$. You showed that (i) is impossible on class prep. Let's show (ii) is impossible.

**6.** By contradiction. Assume (ii) holds: $f(b) > y$. Notice $b \ne a$ because ________________ so $a < b$ by step ________________.

**7.** Let $\epsilon = f(b) - y$. Then $\epsilon > 0$ because ________________________________.

**8.** There is a $\delta > 0 \ni$ if $|x - b| < \delta$, then $|f(x) - f(b)| < \epsilon$ because ________________________________.

**9.** For any $x$ such that $b - \delta < x < b$,

$$f(b) - f(x) \le |f(x) - f(b)| < \epsilon.$$

So $f(x) > y$ because ________________________________. (Hint: What's $\epsilon$?)

**10.** So for any $x$ such that $b - \delta < x < b$, $x \notin S$ because ________________________________.

**11.** This contradicts that ________________________________. So (ii) is false.

**12.** This leaves case (iii) which must be true by Trichotomy. So $b = f(y)$ which is what we want to prove.

**Main Theorem: Justify the Steps for Next Time.** If $f$ is continuous on the closed, bounded interval $[a, b]$, then $f$ is uniformly continuous there.

*Proof*: Given $\epsilon > 0$, we need to find a single value $\delta > 0$ such that if $x$ and $y$ are in $[a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Strategy:

1. use continuity to generate an open cover of $[a, b]$;
2. use the Heine-Borel theorem to obtain a finite subcover of $[a, b]$;
3. use this finiteness to generate the required $\delta$.

**1.** For each $s$ in $[a, b]$ there is a $\delta_s > 0$ such that if $x$ is in $[a, b]$ and $|x - s| < \delta_s$, then $|f(x) - f(s)| < \epsilon/2$ because

____________________________________________. At this point each $\delta$ still depends on $s$.

**2.** For each $s$ in $[a, b]$ form the open interval $O_s = (s - \frac{\delta_s}{2}, s + \frac{\delta_s}{2})$. The collection $\mathcal{O} = \{O_s\}_{s \in [a,b]}$ forms an open cover of $[a, b]$ because

____________________________________________________________

**3.** There is a finite subcollection $C$ in $\mathcal{O}$ that covers $[a, b]$ by ____________________________.
Denote the intervals in $C$ by

$$C = \left\{(s_1 - \tfrac{\delta_1}{2}, s_1 + \tfrac{\delta_1}{2}), \dots, (s_n - \tfrac{\delta_n}{2}, s_n + \tfrac{\delta_n}{2})\right\}.$$

Note the different deltas.

**4.** Let $\delta = \min\{\frac{\delta_1}{2}, \dots, \frac{\delta_n}{2}\}$. Why does this min exist? ________________________________.
Notice that this $\delta$ no longer depends on $s$.

**5.** If $x$ is in $[a, b]$, it is in one of the intervals in $C$, say $(s_k - \frac{\delta_k}{2}, s_k + \frac{\delta_k}{2})$, because ____________________.

**6.** So $|x - s_k| < \frac{\delta_k}{2}$ because ______________________________________________.

**7.** Show: If $y$ is in $[a, b]$ and $|x - y| < \delta$, then $|y - s_k| < \delta + \frac{\delta_k}{2}$. Hint: Use the triangle inequality.

**8.** $\delta + \frac{\delta_k}{2} \le \delta_k$ by step ____________________. So $|y - s_k| < \delta_k$ by step ____________________.

**9.** If $|y - s_k| < \delta_k$, then $|f(y) - f(s_k)| < \epsilon/2$ by step ____________________________________.

Similarly, if $|x - s_k| < \frac{\delta_k}{2} < \delta_k$, then $|f(x) - f(s_k)| < \epsilon/2$ by step __________________________.

**10.** Summarizing: If $|x - y| < \delta$, then $|y - s_k| < \delta_k$ and $|x - s_k| < \delta_k$ by Steps ____________________. So

$|f(x) - f(y)| = |f(x) - f(s_k) + f(s_k) - f(y)| <$ ____________________________ $<$ $\quad$ $<$ $\quad$ .

**11.** So $f$ is uniformly continuous on $[a, b]$. ∎