Math 331 Homework: Day 18

Quote of the Day

> If you open a mathematics paper at random, on the pair of pages before you, you will find a mistake.
> *Joseph L. Doob (1910–2004)*

Practice and Reading

Review Section 2.6 on the IVT and Max-Min Theorem. Begin Chapter 3, reading 3.1. We'll do all of Calculus I in about 5 classes! **Practice Problems**: 2.5.1, 2.5.6, 2.5.8, 2.5.9, 2.6.3, 2.6.5, 2.6.6, 2.6.9, 2.6.11.

1. Show that a linear function $f(x) = mx + b$ is uniformly continuous on the interval $(-\infty, +\infty)$. You may assume that $m \neq 0$.

2. Show that the product of uniformly continuous functions on an interval I need not be uniformly continuous. Hint: Use Example 2.6.1. Consider x^2 on $I = (-\infty, +\infty)$.

3. In the homework due Wednesday, you show that some of the basic limit laws can be extended to one-sided limits: *viz.*: If $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^+} g(x) = M$, then $\lim_{x\to a^+}\big[f(x) + g(x)\big] = L + M$. and $\lim_{x\to a^+} cf(x) = cL$ for any constant c. Similar results hold for products and quotients: $\lim_{x\to a^+}\big[f(x)g(x)\big] = LM$ and $\lim_{x\to a^+}\big[f(x)/g(x)\big] = L/M$ provided $M \neq 0$. Also similar results hold as $x \to a^-$. These rules can now be used to say something about the continuity of a functions on closed intervals. Try proving one or more of these. If f and g are continuous on $[a, b]$, then:

a) $f + g$ is uniformly continuous on $[a, b]$.

b) cf is uniformly continuous on $[a, b]$, where c is any constant.

c) $f \cdot g$ is uniformly continuous on $[a, b]$.

d) If g is never 0 on $[a, b]$, show that f/g is uniformly continuous on $[a, b]$.

4. Prove that $x^{3/2}$ is uniformly continuous on the closed interval $[0, 4]$. Use our earlier work.

5. **Volunteer** to present the Max-Min Theorem next time.

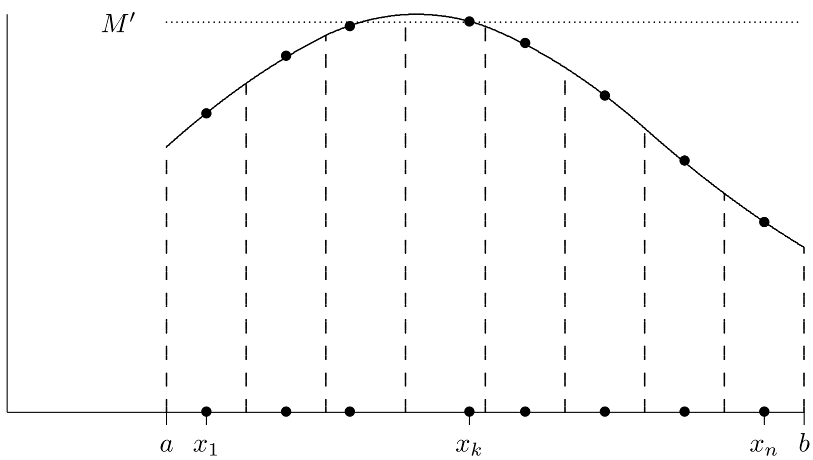
6. **Volunteer** to present Problem 2.6.13 next time otherwise it ends up on the next assignment. Adapt the ideas of Theorem 2.3.4.

The Boundedness Theorem. If f is continuous on the closed interval $[a, b]$, then f is bounded above, i.e., there exists M so that $f(x) \leq M$ for all $x \in [a, b]$. Similarly, f is bounded below, i.e., there exists m so that $m \leq f(x)$ for all $x \in [a, b]$.

Proof: Choose any $\epsilon > 0$ (e.g., $\epsilon = 1$).

1. f is uniformly continuous on $[a, b]$ by ____.

2. So there exists $\delta > 0$ so that if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
$b - a > 0$ and $\delta > 0$ so there exists $n \in \mathbb{N}$ so that $b - a < n\delta$ by ____________________. So $\frac{b-a}{n} < \delta$.
Divide $[a, b]$ into n equal intervals $I_1, \ldots, I_n$ of length $\frac{b-a}{n}$. (See diagram below.)
For each I_k, choose a point $x_k \in I_k$. Let $M' = \max\{f(x_1), \ldots, f(x_n)\}$. (See dots on the curve.)



3. M' may not be an upper bound, but it is close. Let $M = M' + \epsilon$. M does work as an upper bound as we now show:

4. Let $x \in [a, b]$. Then x is in some I_k because ____.

Since the length of I_k is ____________________, then $|x - x_k| < \delta$ because ____________________.

5. So $|f(x) - f(x_k)| < \epsilon$ because ____

6. So $f(x) < f(x_k) + \epsilon$ because ____

7. So $f(x) \leq M' + \epsilon = M$ because ______________________________. So M is an upper bound for f.

8. Give two proof strategies for the sketch of the existence of a lower bound m.