Math 331 Homework: Day 19

Practice, Reading, Volunteer

Reading: Section 3.1 and 3.2. We'll do all of Calculus I in about 5 classes! **Try:** Do both Problems 3.1.10 (a,b, skip c—it is similar to b) AND 3.1.11. **Volunteer** to do Problem 3.1.9 next class.

Test next Friday. **Practice Problems**: 2.5.1 (in fact, uniform!), 2.5.6, 2.5.8, 2.5.9, 2.6.3, 2.6.4 (quick), 2.6.5 (y, y, n; can you give proofs of the first two and a counterexample for the last?), 2.6.6, 2.6.7(a), 2.6.11, and 3.1.8(a)

Hand By Monday at 5pm or EARLIER: You may discuss with a partner—use one copy.

1. Page 88 Problem 2.6.7 (b).

2. You will need the result of this problem in the next problem. Return to the **order** axioms of Chapter 1. Prove: *If* $a > 0$ *and* $n \in \mathbb{N}$, *then* $(a+1)^n > a$. You can do it by induction (justify the inequalities that you use) or you can do it more directly using results that follow from the order axioms.

3. Existence of nth roots. We use functions such as $\sqrt[3]{x}$ all the time. But do roots of real numbers exist? Show that they always do if x is a positive number. Specifically prove: Let $n \in \mathbb{N}$. If a is a **positive** number, then a has an nth root, that is, there is some real number x such that x^n=a. Hint: Let $p(x) = x^n$. Show that $p(0) < a < p(a+1)$. What theorem now applies (make sure to check its hypotheses)?

4. a) Prove: If f is uniformly continuous on the bounded, open interval $I = (a, b)$, then f is bounded above on (a, b). [Hint: Modify the proof of Theorem 2.6.5. A similar proof shows f is bounded below.]

b) In this same situation as part (a), does f necessarily achieve a maximum on I, i.e., must there be a maximum point x_1? Give examples.

5. Let f and g be uniformly continuous on $(-\infty, \infty)$. Prove that the composite $(f \circ g)(x) = f(g(x))$ is uniformly continuous on $(-\infty, \infty)$. Hint: Adapt a proof of a theorem in Section 2.5.

6. Prove: If f and g are uniformly continuous on I and both are bounded,[1] then the product $f \cdot g$ is uniformly continuous on I. Hints: Adapt the ideas in the proof of Theorem 2.3.4 to this setting. How can you use the fact that f and g are bounded to "reduce the proof" to the fact that f and g are uniformly continuous on I? [Aside: This result is NOT true if f and g are not bounded as we saw in class. The linear function $f(x) = g(x) = x$ is uniformly continuous on $(-\infty, +\infty)$. BUT $f(x)g(x) = x^2$ is NOT uniformly continuous on $(-\infty, +\infty)$ as Example 2.6.1 shows. Of course x is not bounded since $\lim_{x\to+\infty} x = +\infty$.]

7. Assume that $f(x)$ is continuous on $(0, 3)$. Each of the following statements is either (a) always true as the consequence of an important theorem or (b) could be false depending on f. If the statement is false provide a counterexample and briefly justify it. If true, then briefly indicate which theorem you are using.

a) f is uniformly continuous on $(0, 3)$.

b) f is bounded on $(0, 3)$.

c) f is uniformly continuous on $[1, 2]$.

d) f is uniformly continuous on $(1, 2)$.

e) f achieves a maximum on the interval $[1, 2]$.

f) f is bounded on $(1, 2)$.

8. Problem 3.1.8(b) without using the product rule. Hint: Factor $(g(x))^n - (g(a))^n$ in a fashion similar to what is done in part (a) of this question in the text.

9. Working with uniform continuity. Let $f(x) = \frac{1}{x}$ throughout this problem.

a) Let b be a finite real number such that $b > 1$. Prove that f is uniformly continuous on the open interval $(1, b)$. See if you can apply some theorems and try to avoid using a proof directly from the definition of uniform continuity. (Think about parts of problem 7.)

b) Prove directly from the ϵ, δ definition of uniform continuity that $f(x)$ is uniformly continuous on $(1, \infty)$. I don't think you can do this using theorems.

c) However, prove that f is NOT uniformly continuous on $(0, \infty)$. Hint: Contradiction?

Over ☞

[1] Remember: f is bounded means that there is a number B so that $|f(x)| \le B$ for all x

10. Fill in the blanks to prove Theorem R below.

Theorem R: Assume that f is continuous on the closed and bounded interval $[a, b]$, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there is a point c strictly between a and b such that $f'(c) = 0$. Prove the theorem by completing the steps below:

a) Proof: f must have maximum and minimum values on $[a, b]$ by ______________________________.

b) There are two possibilities: (i) **both** of these extreme values occur at the two endpoints a and b of $[a, b]$ or (ii) at least one of the extreme values occurs at a point c (located where?)

c) In case (i), prove that f must be constant on $[a, b]$ by using the definitions of maximum and minimum value.

d) Since f is constant, then for any point c between a and b, $f'(c) =$ _____________, so the theorem holds.

e) In case (ii) f has an extreme point c (located where?) ____________________________________.

At any such extreme point $f'(c) =$ __________ by (state the appropriate theorem and show it applies).