Math 331: Day 21–The MVT

We will reverse the order of topics and discuss the Mean Value Theorem today and return to the derivation of the more complicated derivative rules next time. In what follows, we will also assume that the familiar constant multiple, sum, and difference rules hold for derivatives. We will discuss them next time.

1. First a quick result: If $f(x) = mx + b$, then $f'(x) = m$ for all x. In particular, this means If $f(x) = b$ is a constant function then $f'(x) = 0$ or if $f(x) = x$, then $f'(x) = 1$.

Proof:

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} =$

Theorem R states: Assume that f is continuous on the closed and bounded interval $[a, b]$, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there is a point c strictly between a and b such that $f'(c) = 0$.

The Mean Value Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2. Use Theorem R to carefully prove the Mean Value Theorem.

3. **First Consequences of the MVT**. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Under these hypotheses:
   a) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
   b) If $f'(x) > 0$ for all $x \in (a, b)$ and if $x, y \in (a, b)$ with $x < y$, then $f(x) < f(y)$ (that is, f is **increasing** on $[a, b]$.
   c) If $f'(x) < 0$ for all $x \in (a, b)$ and if $x, y \in (a, b)$ with $x < y$, then $f(x) > f(y)$ (that is, f is **decreasing** on $[a, b]$.
   d) If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is one-to-one on $[a, b]$.
   e) Additionally, if g is continuous on $[a, b]$, differentiable on (a, b) with $g'(x) = f'(x)$ for all $x \in (b,)$, then there is a constant k so that $g(x) = f(x) + k$ on $[a, b]$.

4. EZ. Use the MVT to prove: If $b \geq 0$, then $\sin b \leq b$. (Assume Calculus I knowledge.) Hint: The result is clearly true if $x = 0$ (right?). So assume $b > 0$. Let $f(x) = \sin x$ on $[0, b]$.

5. Use the MVT to prove **Bernoulli's Inequality**: For all $b > 0$ and for all $n \in \mathbb{N}$,
$$(1 + b)^n \geq 1 + nb.$$
   What's $f(x)$ this time? Note: This can be done by induction, but it is quicker with the MVT.

6. Prove: If $b > 1$, then $\dfrac{b}{b-1} < \ln b < b - 1$.

Differentiability on Closed Intervals. g is differentiable on the closed interval $[a, b]$ if g is differentiable at each point in the open interval (a, b) and the appropriate one-sided derivatives exist at a and b. Specifically

1. g is differentiable at each $x \in (a, b)$,
2. $\displaystyle\lim_{x \to a^+} \frac{g(x) - g(a)}{x - a}$ exists (and is denoted by $g'(a)$), and $\displaystyle\lim_{x \to b^-} \frac{g(x) - g(b)}{x - b}$ exists (and is denoted by $g'(b)$).

7. **Intermediate Value Theorem for Derivatives.** If f is differentiable on $[a, b]$ and $f'(a) < k < f'(b)$, then there is a $c \in (a, b)$ with $f'(c) = k$. [A similar result holds if $f'(a) > k > f'(b)$.]
   a) Consider the auxiliary function: For $x \in [a, b]$ define $g(x) = f(x) - kx$. Show that g is differentiable on $[a, b]$ and that $g'(a) < 0 < g'(b)$.
   b) Prove that g has a minimum point $c \in [a, b]$.
   c) From part (a), $0 < g'(b) = \displaystyle\lim_{x \to b^-} \frac{g(x) - g(b)}{x - b}$. Prove that there exists $\delta > 0$ so that if $-\delta < x - b < 0$, then $0 < \frac{g(x)-g(b)}{x-b}$. Hint: Let $\epsilon = g'(b)$.
   d) With this same δ prove: If $-\delta < x - b < 0$, then $g(x) < g(b)$. [This shows that $g(b)$ is NOT the minimum value of g. A similar argument shows that $g(a)$ is also not the minimum value of g. In other words, $c \neq a$ and $c \neq b$.]
   e) So $c \in (a, b)$. Prove $g'(c) = 0$ and then show $f'(c) = k$.

8. True or False: The Dirichlet function $D(x)$ is the derivative of some function $F(x)$ on the interval $[a, b]$. (Is $D(x) = F'(x)$ for some function F?) Explain.

9. **Corollary of IVTFD.** If f is differentiable on $[a, b]$ and $f'(x) \neq 0$ for all $x \in (a, b)$, then either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$. (So from Problem 1, f is either always increasing or always decreasing on $[a, b]$.)

10. **The Cauchy Mean Value Theorem.** Suppose that f and g are continuous on the closed, bounded interval $[a, b]$ and are differentiable on (a, b). Then there is a point c strictly between a and b such that
$$\big(g(b) - g(a)\big)f'(c) = \big(f(b) - f(a)\big)g'(c).$$
   a) Define the auxiliary function $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$. Show that h is continuous on $[a, b]$ and differentiable on (a, b).
   b) Show that $h(a) = h(b)$.
   c) Apply the mean value theorem to h and show that c is the desired point.