Math 331 Homework, Day 24

Problems 1 through 4 and 6 (5 is optional) are due Wednesday. All are short and their purpose is to get you to review before we start on integration on Wednesday. **Read** Section 3.4 up to page 122.

1. Use an ϵ, δ proof to show that $\lim_{x \to 5} \frac{x+5}{x^2 - x} = \frac{1}{2}$. Be careful of signs when factoring and bounding.

2. Listed below are various properties of functions.

1. f is continuous on $[-2, 2]$.
2. f is differentiable on $[-2, 2]$.
3. f is uniformly continuous on $[-2, 2]$.
4. f is bounded on $[-2, 2]$.

For each of the following find a function which satisfies the required combination. If such a combination is impossible briefly justify why it is impossible.

a) A function satisfying 1 but not 2.
b) A function satisfying 2 but not 3.
c) A function satisfying 1 but not 3.
d) A function satisfying 2 but not 4.
e) A function satisfying 4 but not 1.

3. Let $f(x)$ be a continuous function on a closed, bounded interval $[a, b]$, Suppose that $f(x) > 0$ for all $x \in [a, b]$. Then there is an $\epsilon > 0$ such that $f(x) > \epsilon$ for all $x \in [a, b]$. Hint: Apply a theorem; justify why it applies.

4. Prove this weaker version much more simply from scratch without using the Generalized Mean Value Theorem (or even the MVT). Hint: Use the definition of the derivative! It should be quick.

L'Hôpital's Rule (Weak Version): Assume that f and g are differentiable on an open interval I containing a. If $f(a) = 0$ and $g(a) = 0$ but $g'(a) \neq 0$. then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

5. We mentioned in class that **L'Hôpital's Rule extends to limits at infinity.** That is:

Assume that f and g are differentiable on the interval $(a, +\infty)$ and assume that

$$\lim_{x \to +\infty} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} g(x) = 0.$$

Further, assume that $g'(x) \neq 0$ for all $x \in (a, +\infty)$. If $\lim_{x \to +\infty} f'(x)/g'(x)$ exists, then

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}.$$

An analogous result holds for limits as $x \to -\infty$.

This extension to infinity depends on Theorem 2.4.6 that relates one-sided limits to limits at infinity.

Theorem 2.4.6: Assume that f is a function defined on an interval of the form $(a, +\infty)$. Let $y = 1/x$. Then

$$\lim_{x \to +\infty} f(x) = L \iff \lim_{y \to 0^+} f(\tfrac{1}{y}) = L.$$

An analogous result holds for limits as $x \to -\infty$.

a) **Extra Credit:** This theorem shows that we can compute limits at infinity by computing one-sided limits at 0 where we know L'Hôpital's Rule applies. So here's the bonus question: Prove Theorem 2.4.6 $\Leftarrow$. The other direction is done in your text. This is a good exercise in using the limit definitions.

The Mean Value Theorem. Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there is a point c strictly between a and b such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

First Consequences of the MVT. We have already proven the following: Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Under these hypotheses

- **a)** If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$. (Theorem 3.3.4, done in class.)
- **b)** If $f'(x) > 0$ for all $x \in (a,b)$ and if $x, y \in (a,b)$ with $x < y$, then $f(x) < f(y)$ (that is, f is **increasing** on $[a,b]$. (Done in class.)
- **c)** If $f'(x) < 0$ for all $x \in (a,b)$ and if $x, y \in (a,b)$ with $x < y$, then $f(x) > f(y)$ (that is, f is **decreasing** on $[a,b]$. (Similar to above.)
- **d)** If $f'(x) \neq 0$ for all $x \in (a,b)$, then f is one-to-one on $[a,b]$. (On Exam 2.)

6. Use (a) to prove: If f and g are continuous on $[a,b]$, differentiable on (a,b) with $g'(x) = f'(x)$ for all $x \in (a,b)$, then there is a constant k so that $g(x) = f(x) + k$ on $[a,b]$.

Differentiability on Closed Intervals. g is differentiable on the closed interval $[a,b]$ if g is differentiable at each point in the open interval (a,b) and the appropriate one-sided derivatives exist at a and b. Specifically

1. g is differentiable at each $x \in (a,b)$,
2. $\lim_{x\to a^+} \frac{g(x)-g(a)}{x-a}$ exists (and is denoted by $g'(a)$), and $\lim_{x\to b^-} \frac{g(x)-g(b)}{x-b}$ exists (and is denoted by $g'(b)$).

Note: All basic derivative rules (e.g., sum, product) carry over to functions differentiable on closed intervals.

OTHER CONSEQUENCES, #7,8 on next Assignment

7. Intermediate Value Theorem for Derivatives. If f is differentiable on $[a,b]$ and $f'(a) < k < f'(b)$, then there is a $c \in (a,b)$ with $f'(c) = k$. A similar result holds if $f'(a) > k > f'(b)$. (Note: We cannot apply the IVT because we do not know that f' is continuous on $[a,b]$.)

- **a)** Consider the auxiliary function $g(x) = f(x) - kx$, for $x \in [a,b]$. Since f and x are differentiable on $[a,b]$ it follows that g is differentiable on $[a,b]$. Show that $g'(a) < 0 < g'(b)$.
- **b)** Prove that g has a minimum point $c \in [a,b]$.
- **c)** From part (a), $0 < g'(b) = \lim_{x\to b^-} \frac{g(x)-g(b)}{x-b}$. Use the definition of one-sided limit to prove that there exists $\delta > 0$ so that if $-\delta < x - b < 0$, then $0 < \frac{g(x)-g(b)}{x-b}$. Hint: Let $\epsilon = g'(b)$.
- **d)** With this same δ prove: If $-\delta < x - b < 0$, then $g(x) < g(b)$. [This shows that $g(b)$ is NOT the minimum value of g. A similar argument shows that $g(a)$ is also not the minimum value of g. In other words, $c \neq a$ and $c \neq b$.]
- **e)** So $c \in (a,b)$. Prove $g'(c) = 0$ and then show $f'(c) = k$.

8. True or False: The Dirichlet function $D(x)$ is the derivative of some function $F(x)$ on the interval $[a,b]$. (Is $D(x) = F'(x)$ for some function F?) Explain.

9. Corollary of IVTFD. If f is differentiable on $[a,b]$ and $f'(x) \neq 0$ for all $x \in (a,b)$, then either $f'(x) \geq 0$ for all $x \in [a,b]$ or $f'(x) \leq 0$ for all $x \in [a,b]$. (So from Problem 1, f is either always increasing or always decreasing on $[a,b]$.)

10. The Cauchy Mean Value Theorem. Suppose that f and g are continuous on the closed, bounded interval $[a,b]$ and are differentiable on (a,b). Then there is a point c strictly between a and b such that

$$\big(g(b)-g(a)\big)f'(c) = \big(f(b)-f(a)\big)g'(c).$$

- **a)** Define the auxiliary function $h(x) = (g(b)-g(a))f(x) - (f(b)-f(a))g(x)$. Show that h is continuous on $[a,b]$ and differentiable on (a,b).
- **b)** Show that $h(a) = h(b)$.
- **c)** Apply the mean value theorem to h and show that c is the desired point.