Math 331 Homework: Day 25

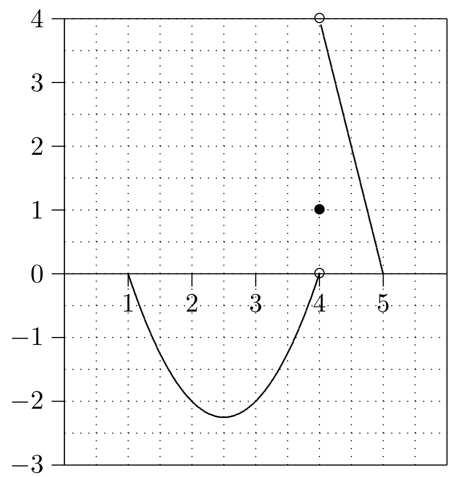
Make sure you have read Section 3.4. Begin reading Section 3.5. Make sure you can define all the key terms (and are comfortable with the corresponding notation) such as partition, lower and upper sums, infs, sups, etc.

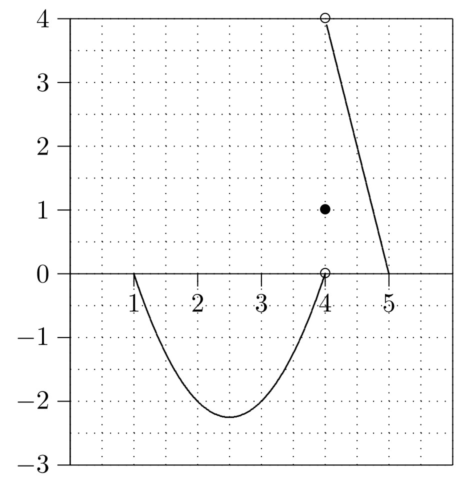
Hand in on Wednesday

0. See/Do the assigned presentation problems on the back.

1. The definition of the derivative in Math 130 or Math 331 is (exactly) the same. However, things are different with integration. Even though there are more complex approaches to integration (using measure theory), the approach we have taken is much more sophisticated than in Math 131. Find your old calculus textbook (or someone else's). Look up and copy out its definition of the **definite integral** (not anti-dfferentiation). What is the major difference in your view?

2. Let f be the function on the interval $[1, 5]$ graphed below. Note: On the interval $[1, 4)]$ the equation is $f(x) = x^2 - 5x + 4$. You should be able to figure out the equation of f at $x = 4$ and on $(4, 5]$. Let $P = \{1, 3/2, 2, 3, 4, 9/2, 5\}$. Determine the exact values of $U(P)$ and $L(P)$. You may wish to draw $U(P)$ and $L(P)$.





3. Modify the Dirichlet function as follows: Define f on $[0, 1]$ by $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

- **a)** Prove: If $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[0, 1]$, then $U(P, f) > \frac{1}{2}$. Hint: The midpoint of the ith subinterval is $\frac{x_i + x_{i-1}}{2}$. Use density to show that $M_i > \frac{x_i + x_{i-1}}{2}$. Also use the ideas at the end of Example 3.4.3 to evaluate $U(P, f)$.
- **b)** Prove: If $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[0, 1]$, then $L(P, f) = 0$.
- **c)** Prove f is not integrable on $[0, 1]$.

4. Give an example of a function f that is defined and bounded but not integrable on $[0, 1]$, however f^2 is integrable. Hint: Adapt (modify) the definition of the Dirichlet function $D(x)$ to create a function whose square is a constant function. Use upper and lower sums to indicate why your function f is not integrable.

5. In this problem assume f to be continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Earlier in the term we proved a lemma about such a function: If $c \in [a, b]$, and $f(c) > 0$, then there is a subinterval $[x_1, x_2] \subset [a, b]$ so that for all $x \in [x_1, x_2]$, $f(x) > 0$. Now use this lemma to prove: If $\sup_P \{L(P, f)\} = 0$, then $f(x) = 0$ for all $x \in [a, b]$. [Hint: By contradiction. If $f(c) > 0$, use a partition based on the lemma.]

6. Assume that f is integrable on $[a, b]$. Suppose that J is a real number such that $L(P, f) \leq J \leq U(P, f)$ for every partition P of $[a, b]$. Show that $J = \int_a^b f$. (Hint: Use Theorem 3.4.9, presentation problem 1, and Problem 1.3.19.)

Differentiability on Closed Intervals. This definition is used in the next problem: We say g is **differentiable on the closed interval** $[a,b]$ if g is differentiable at each point in the open interval (a,b) and the appropriate one-sided derivatives exist at a and b. Specifically

1. g is differentiable at each $x \in (a,b)$,
2. $\lim\limits_{x\to a^+} \dfrac{g(x)-g(a)}{x-a}$ exists (and is denoted by $g'(a)$), and $\lim\limits_{x\to b^-} \dfrac{g(x)-g(b)}{x-b}$ exists (and is denoted by $g'(b)$).

Note: All basic derivative rules (e.g., sum, product) carry over to functions differentiable on closed intervals.

7. Along with the IVT, MVT, and the Extreme Value (Max-Min) Theorem, this next result is one of the "four pillars of elementary analysis." **Darboux's Theorem (Intermediate Value Theorem for Derivatives).** If f is differentiable on $[a,b]$ and $f'(a) < k < f'(b)$, then there is a $c \in (a,b)$ with $f'(c) = k$. A similar result holds if $f'(a) > k > f'(b)$. (Note: While it looks like we should be able to use the IVT, we cannot because we do not know that f' is continuous on $[a,b]$.)

- **a)** Consider the auxiliary function $g(x) = f(x) - kx$, for $x \in [a,b]$. Since f and kx are differentiable on $[a,b]$, it follows that their difference g is differentiable on $[a,b]$. Prove that g has a minimum point $c \in [a,b]$.
- **b)** Check that $g'(a) < 0 < g'(b)$.
- **c)** From part (b), $0 < g'(b) = \lim\limits_{x\to b^-} \dfrac{g(x)-g(b)}{x-b}$. Use the definition of one-sided limit to prove that there exists $\delta > 0$ so that if $-\delta < x - b < 0$, then $0 < \frac{g(x)-g(b)}{x-b}$. Hint: Let $\epsilon = g'(b)$.
- **d)** With this same δ prove: If $-\delta < x - b < 0$, then $g(x) < g(b)$. [This shows that $g(b)$ is NOT the minimum value of g. A similar argument shows that $g(a)$ is also not the minimum value of g. In other words, $c \neq a$ and $c \neq b$.]
- **e)** So $c \in (a,b)$. Prove $g'(c) = 0$ and then show $f'(c) = k$.

8. True or False: The Dirichlet function $D(x)$ is the derivative of some function $F(x)$ on the interval $[a,b]$. (Is $D(x) = F'(x)$ for some function F?) Explain. Consider the previous problem.

Present on Friday (?) or Monday

1. [Kelcie—Friday] For some students, this next result is a puzzle. Prove the following: Let $a, b \in \mathbb{R}$. If for every $\epsilon > 0$, we have $|b - a| < \epsilon$, then $a = b$. Hint: Contradiction.

2. [Abby—Friday] Problem 3.4.2 (page 128). Prove: Let f be a bounded function on $[a,b]$. If Q is a refinement of a partition P, then $0 \le U(Q) - L(Q) \le U(P) - L(P)$. (Hint: Use Thm 3.4.4 and Problem 1.3.19.)

3. **Definition.** A function f defined on an interval I is *non-decreasing* if whenever $x_1, x_2 \in I$ and $x_1 < x_2$, then $f(x_1) \le f(x_2)$.

- **a)** [Stuart and Tianrui—Monday] Show that if f is a bounded, *non-decreasing* function on $[a,b]$ and $P_n = \{x_0, x_1, \ldots, x_n\}$ is the partition of $[a,b]$ into n equal width subintervals of length $x_i - x_{i-1} = \frac{(b-a)}{n}$, then
$$U(P_n) - L(P_n) = [f(b) - f(a)] \cdot \frac{(b-a)}{n}.$$
- **b)** Then use Fact 5 (Theorem 3.4.9) to prove that f is integrable on $[a,b]$.

4. [Aysmel—Monday] Suppose that f is differentiable on $[a,b]$ and that $f'(x) > 0$ for all $x \in [a,b]$. Prove that f is integrable on $[a,b]$.

5. [Ivy and Ryo—Monday] Prove: If f is integrable on $[a,b]$, then $-f$ is integrable on $[a,b]$ using the following steps:

- **a)** Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$. Let $M_i = \sup\{f(x) \mid x \in [x_{i_1}, x_i]\}$. What problem in Chapter 1 shows that $-M_i = \inf\{-f(x) \mid x \in [x_{i_1}, x_i]\}$? (No need to reprove it.)
- **b)** Using part (a) prove that $L(P, -f) = -U(P, f)$. (Similarly one can show $U(P, -f) = -L(P, f)$.)
- **c)** Given any $\epsilon > 0$. Prove that there is a partition Q of $[a,b]$ so that $U(Q, -f) - L(Q, -f) < \epsilon$ and conclude that $-f$ is integrable. Hint: Use the fact that f is integrable and Theorem 3.4.9.

6. **a)** [Caroline and Richard—Monday] Define what it means for a function to be *non-increasing* on an interval I.

- **b)** Then prove: If f is non-increasing on $[a,b]$, then f is integrable on $[a,b]$. Hint: Consider $-f$ and two of the previous problems to do this very quickly.