Math 331 Homework: Day 27

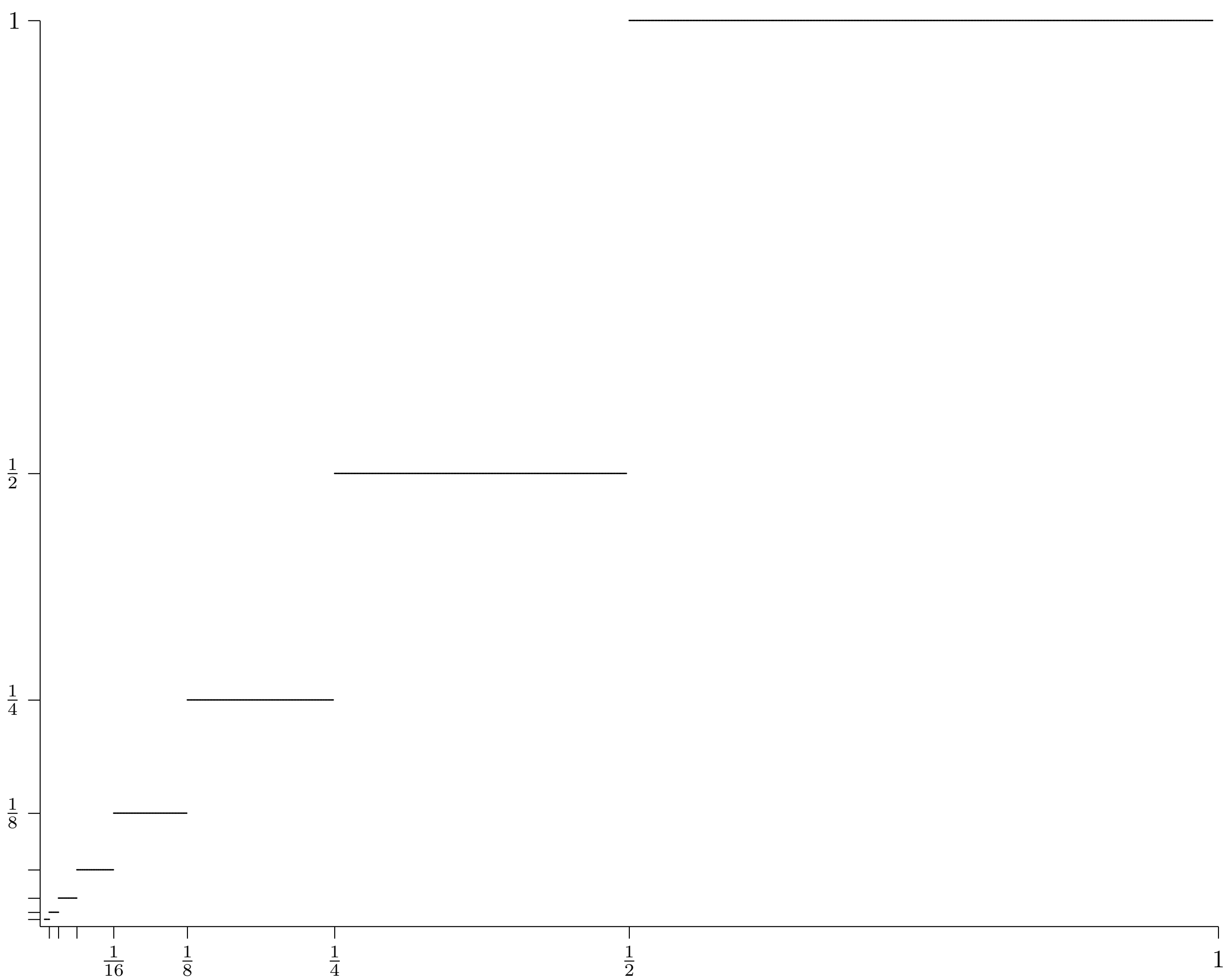
Reading: You should be reading in Section 3.5 and 3.6 for next week. Review Section 3.4 as necessary. Make sure you can define all the key terms (and are comfortable with the corresponding notation) such as partition, lower and upper sums, infs, sups, etc.

EXTRA CREDIT Opportunities: Volunteer to Present

See Page 2 for two opportunities to present (with a friend) results for Extra Credit on Friday. **Everyone look at Page 3 for next class**, try to fill in the blanks, and be prepared to discuss the proof. (It is not in the text.) Someone lead us through it for Extra Credit. Finally, if we don't get to the example below today, someone should present the answers next time for extra credit. See me for help or to volunteer.

An Integrable Function with an Infinite Number of Discontinuities. A function may be highly discontinuous and yet still be integrable. Define the step function f on $[0,1]$ as follows (see graph). For any natural number n, let

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2^n}, & \text{if } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}. \end{cases}$$



1. A major theorem that we will prove is: `If` f `is continuous on` $[a,b]$`, then` f `is integrable on` $[a,b]$.

2. Observe that the function f graphed above has an infinite number of discontinuities. They occur at that occur at the points $1/2^n$, where $n \in \mathbb{N}$. So we can't use continuity to show f is integrable. What fact does apply to show that f is integrable?

3. Give a "proof" that $\displaystyle\int_0^1 f(x)\,dx = \frac{2}{3}$.

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4. (2 Volunteers) Page 131: #3.4.5 (a and b, and probably (c)). Show that the "skyscraper" function (graph it),

$$s(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \leq 1, \end{cases}$$

is integrable on $[0, 1]$ and that $\int_0^1 s(x)\,dx = 0$. Use the following process.

a) For $n \in \mathbb{N}$ let P_n be the partition $\{0, \frac{1}{n}, 1\}$. Compute $L(P_n, s)$ and $U(P_n, s)$.

b) Show that s is integrable by showing that for any $\epsilon > 0$, there is a sufficiently large $n \in \mathbb{N}$ such that $U(P_n, s) - L(P_n, s) < \epsilon$.

c) Show that $\int_0^1 s = 0$.

5. (2 Volunteers) Suppose that f is integrable on $[a, b]$.

a) Then by definition f must be bounded on $[a, b]$. In particular, suppose that $m \leq f(x) \leq M$ for all $x \in [a, b]$, where m and M are *any* lower and upper bounds for f on $[a, b]$. Show that $m(b - a) \leq \int_a^b f \leq M(b - a)$.

b) Show: If f is nonnegative and integrable on $[a, b]$, then $0 \leq \int_a^b f$. Use part (a).

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Additivity. The second part of the **additivity theorem** (Theorem 3.5.3) states that *if $a < b < c$ and f is integrable over $[a, c]$, then f is integrable over $[a, b]$ and $[b, c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$.*

The converse of this statement is the first part of the theorem and it is proved in the text. Complete the proof of the statement above by answering these questions.

a) Proof: Start by taking $\epsilon > 0$. $\exists$ a partition P of $[a, c]$ such that $U(P) - L(P) < \epsilon$ by ____________________.

b) Let Q be the **refinement** of P obtained by adjoining the point b to P. Denote the points in Q by $\{x_0, \ldots, x_k = b, \ldots, x_n\}$. $x_k = b$ is the point we added in. (This is why even if $P = Q$ we may want to say that Q is finer than P. If b is already one of the division points of P, then $P = Q$. But the conclusion of the Fact 2 [Refinement] is still true.) That is, since Q is finer than P, then

$$L(P) \leq L(Q) \leq U(Q) \leq U(P).$$

As Abby showed (and using part (a)) this means that $U(Q) - L(Q) \leq U(P) - L(P) <$ ____________________.

c) Split the partition Q into two pieces $Q_1 = \{a = x_0, \ldots, x_k = b\}$ and $Q_2 = \{b = x_k, \ldots, x_n = c\}$ to obtain partitions of $[a, b]$ and $[b, c]$, respectively. Fill in the lower and upper indices on the three lower summation signs:

$$L(Q_1, [a, b]) = \sum_{i=} m_i(x_i - x_{i-1}) \qquad L(Q_2, [b, c]) = \sum_{i=} m_i(x_i - x_{i-1}) \qquad L(Q, [a, c]) = \sum_{i=} m_i(x_i - x_{i-1})$$

d) Use (c) to fill in the missing mathematical symbols: $L(Q_1, [a, b])$ __________ $L(Q_2, [b, c])$ __________ $L(Q, [a, c])$.

e) Similarly: $U(Q_1, [a, b])$ ____________ $U(Q_2, [b, c])$ ____________ $U(Q, [a, c])$.

f) Rearrange (d) and (e) to get

$$\big(U(Q_1, [a, b]) - L(Q_1, [a, b])\big) + \big(U(Q_2, [b, c]) - L(Q_2, [b, c])\big) = \underline{\qquad\qquad\qquad}.$$

g) Explain using previous steps why $U(Q_1, [a, b]) - L(Q_1, [a, b]) < \epsilon$.

h) So f is integrable on $[a, b]$ by ____________________________________. Similarly f is integrable on $[b, c]$.

i) Notice that the hypothesis of the first half of the additivity theorem is now satisfied. So $\int_a^c f = \int_a^b f + \int_b^c f$.