Math 331 Homework: Day 28

Finish reading Section 3.5 and begin Section 3.6.

1. **Volunteer For Friday**: Be prepared to discuss/ask questions on the material on the back of this page. Fill in the blanks. Similarly for page 3: Fill in as many blanks as you can for the Composition theorem.

Hand in on Wednesday

1. There are several ways to prove **The Square Theorem:** *If f is integrable on $[a,b]$, then f^2 is also integrable on $[a,b]$.* I want you to use the following method: First review the Sup Lemma and the ideas in the proof of the Absolute Value Theorem on the Day 26 Handout. Now:
   a) Since f is integrable f is bounded on $[a,b]$. Show that there exists $K > 0$ so that for all $x, y \in [a,b]$, $|f(x)+f(y)| < K$.
   b) Given $\epsilon > 0$. Since f is integrable, explain why there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon/K$.
   c) Show that $K[M_i(f) - m_i(f)]$ is an upper bound for the set $\{|(f(x))^2 - (f(y))^2| : x, y \in [x_{i-1}, x-i]\}$.
   d) Use the Sup Lemma to prove
$$U(P,f^2) - L(P,f^2) \leq \sum_{i=1}^{n} K\left[M_i(f) - m_i(f)\right](x_i - x_{i-1})$$
   e) Show $U(P,f^2) - L(P,f^2) < \epsilon$ and finish the proof.

2. Use the Composition Theorem (see page 3) to prove the following. Explicitly indicate how the hypotheses of the Composition Theorem are satisfied. What is the function g in each case?
   a) Suppose that $f(x)$ is integrable on $[a,b]$. Prove: If $n \in \mathbb{N}$ then $f^n(x)$ is integrable on $[a,b]$. [Note: $f^n(x)$ is just $(f(x))^n$. In particular, this gives us another proof of the Square Theorem: f^2 is integrable whenever f is.]
   b) Suppose that f is integrable on $[a,b]$ and that there exists $k > 0$ so that $f(x) \geq k$ for all $x \in [a,b]$. Prove that $1/f$ is integrable on $[a,b]$.

3. Page 137 #3.5.1. Bonus for giving more than one (correct) reason.

4. This result is very useful, including in the proof of the Second Fundamental Theorem of Calculus. Page 137 Problem 3.5.3. Hint: For part (b) recall Volunteer Problem #2 on Day 27 Handout.

5. This problem is important in leading up to the proof of the First Fundamental Theorem of Calculus. Prove: *If f is integrable on $[a,b]$, then* $\left|\int_a^b f\right| \leq \int_a^b |f|$. Use the following steps
   a) First, what theorem guarantees that $|f|$ is integrable?
   b) Second, since $|f(x)| \leq |f(x)|$, by Theorem 1.3.12, $-|f(x)| \leq f(x) \leq |f(x)|$. Now use the previous problem and Theorem 1.3.12 again to finish the proof.

6. (Integral Squeeze Theorem.) *Suppose that f and h are integrable on $[a,b]$ and suppose further that g is defined on $[a,b]$ with $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$. If $\int_a^b f = \int_a^b h$, then g is integrable on $[a,b]$ and $\int_a^b g = \int_a^b f$.* (Hints: Prove that there is a partition Q of $[a,b]$ so that $-\epsilon/2 + \int_a^b f < L(Q,f)$ and similarly a partition R of $[a,b]$ so that $U(R,h) < \epsilon/2 + \int_a^b h$. Use a refinement and compare the upper and lower sums to those for g and use Theorem 3.4.9.

7. Page 138 #3.5.7. Hint: What's the only function you know right now that is not integrable?

8. This is a refinement of Problem 3.4.4 which is an Extra Credit presentation problem #5 from Day 27. Suppose that g is *continuous* and *positive* on the closed bounded interval $[a,b]$. Prove that $0 < \int_a^b g$. Note that the inequality is strict. Hint: Remember that g is *continuous* on the closed bounded interval $[a,b]$. What theorems from an earlier chapter apply? Use one of them to say something intelligent about $m = \inf\{g(x) \mid a \leq x \leq b\}$. Then use the presentation problem from Day 27.

Volunteer to Present for Extra Credit

Functions that are continuous except at a finite number of points. This problem combines the ideas in the proof of the Additivity of Intervals Theorem and the problem on the skyscraper function. We will show that if a bounded function f is continuous except at a *finite* number of points on $[a, b]$, then f is integrable. The heart of the matter is settled in the following result:

> **Lemma:** *Assume f is bounded on $[a, b]$ and continuous on $[a, b]$ except at a. Then f is integrable on $[a, b]$. (A similar result holds if f is bounded on $[a, b]$ and continuous on $[a, b]$ except at b.* Combining these two results using additivity of intervals we get the **Lemma Extension**: *If f is bounded on $[a, b]$ and continuous on $[a, b]$ except at some point c between a and b, then f is integrable on $[a, b]$.)*

a) We will eventually use Theorem 3.4.9. So suppose $\epsilon > 0$. Next, f has a supremum M and an infimum m on $[a, b]$ because f is a ____________________________________ function.

b) Next, to prove the lemma begin by enclosing the discontinuity at a in a small subinterval $[a, a + \frac{1}{n}]$, where by the Archimedean Principle we can choose $n \in \mathbb{N}$ such that $(M - m)\frac{1}{n} < \epsilon/2$ and $a + \frac{1}{n} < b$. Let $x_1 = a + \frac{1}{n}$. Since f is ____ on $[x_1, b]$, then f is integrable on $[x_1, b]$.

c) By Theorem ____________________ there is a partition Q of $[x_1, b]$ so that $U(Q, [x_1, b]) - L(Q, [x_1, b]) < \epsilon/2$.

d) Now return to the subinterval $[a, a + \frac{1}{n}] = [a, x_1]$. Let $m_1 = \inf\{f(x)|x \in [a, x_1]\}$ and $M_1 = \sup\{f(x)|x \in [a, x_1]\}$. Prove that $M_1 - m_1 \leq M - m$. Hint: Use problem 1.3.19.

e) It now follows that $(M_1 - m_1)(x_1 - a) \leq (M - m)(x_1 - a) = (M - m)\frac{1}{n} < \epsilon/2$ (use Theorem 1.3.9, part d, and part b).

f) Let $P = \{a, a + \frac{1}{n} = x_1, x_2, \ldots, b\}$ be the partition of $[a, b]$ obtained from the union of the single point a with the points of the partition Q of $[x_1, b]$. Just as in the proof of the additivity theorem, we can split the partition and the corresponding sums into two pieces:

$$L(P, [a, b]) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + \sum_{i=2}^{n} m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + L(Q, [x_1, b])$$

$$U(P, [a, b]) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + \sum_{i=2}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + U(Q, [x_1, b]).$$

Use Theorem 3.4.9 to complete the proof of the lemma.

Think about how you could use the Lemma and its extension to show that the theorem holds if there are n discontinuities in $[a, b]$. Draw a picture to illustrate your idea.

Theorem (Composition and Integrability). Suppose that f is integrable on $[a,b]$ and that $c \leq f(x) \leq d$ for all $x \in [a,b]$.[1] Assume further that g is continuous on $[c,d]$. Then the composite $g \circ f$ is integrable on $[a,b]$.

Proof: Why would this proof be easy if both f and g were continuous? The proof is a bit complicated notationally. We will use Theorem 3.4.9, so let $\epsilon > 0$. (Review the Sup Lemma (Day 26 Handout) before continuing.)

a) Let $K = \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\}$. Why does K exist?

b) Choose $\epsilon' = \frac{b-a+K}{\epsilon} > 0$. (We'll see why later.) g is uniformly continuous on $[c,d]$ by ________________________________

c) So there is a $\delta' > 0$ so that whenever $s,t \in [c,d]$ and $|s-t| < \delta'$, then ________________________________ $< \epsilon'$. [And for technical reasons, we will want to choose $\delta < \epsilon'$. So let $\delta = \min\{\delta', \epsilon'\}$.]

d) Next, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \delta^2$ by ________________.

e) Now we will show that

$$U(P, g \circ f) - L(P, g \circ f) < \sum_{i=1}^{n} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) < \epsilon.$$

To do this, we separate the set of indices of the partition P into two disjoint sets. On the first set we make $M_i(g \circ f) - m_i(g \circ f)$ small and on the second set we make $\sum(x_i - x_{i-1})$ small. Let

$$A = \{i : M_i(f) - m_i(f) < \delta\} \qquad \text{and} \qquad B = \{i : M_i(f) - m_i(f) \geq \delta\}.$$

If $i \in A$ and $x, y \in [x_{i-1}, x_i]$, then explain why:

$$|f(x) - f(y)| \leq M_i(f) - m_i(f) < \delta.$$

f) So if $x, y \in [x_{i-1}, x_i]$, then $|(g \circ f)(x) - (g \circ f)(y)| = |g(f(x)) - g(f(y))| < \epsilon'$ by Step ____________.

g) So $M_i(g \circ f) - m_i(g \circ f) \leq \epsilon'$ by ________________________________.

h) Adding we get (justify the three inequalities)

$$\sum_{i \in A} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) \leq \sum_{i \in A} \epsilon'(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \epsilon'(x_i - x_{i-1}) \leq \epsilon'(b-a)$$

i) What if $i \in B$? Then $\frac{M_i(f) - m_i(f)}{\delta} \geq 1$ because ________________________________. So (justify each inequality)

$$\sum_{i \in B} 1(x_i - x_{i-1}) \leq \sum_{i \in B} \left(\frac{M_i(f) - m_i(f)}{\delta}\right)(x_i - x_{i-1}) \leq \sum_{\text{all } i} \left(\frac{M_i(f) - m_i(f)}{\delta}\right)(x_i - x_{i-1}) = \frac{U(P,f) - L(P,f)}{\delta} < \delta < \epsilon'.$$

j) By part (a): $M_i(g \circ f) - m_i(g \circ f) \leq \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\} = K$ so (justify)

$$\sum_{i \in B} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) \leq \sum_{i \in B} K(x_i - x_{i-1}) = K \sum_{i \in B} 1(x_i - x_{i-1}) \leq K\epsilon'.$$

k) Now recombine all the indices:

$$\begin{aligned} U(P, g \circ f) - L(P, g \circ f) &= \sum_{i \in A} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) + \sum_{i \in B} [M_i(g \circ f) - m_i(g \circ f)](x_i - x_{i-1}) \\ &\leq \epsilon'(b-a) + K\epsilon' = \underline{\hspace{3cm}} < \underline{\hspace{1.5cm}}. \end{aligned}$$

So $g \circ f$ is integrable by ____________________.

[1] The usual notation for $c \leq f(x) \leq d$ for all $x \in [a,b]$ is to say: $f([a,b]) \subset [c,d]$.