Math 331 Homework: Day 29

Read/Review Section 3.5 which we are close to finishing. Start reading Section 3.6 which we will start on Monday. Pay particular attention to Theorems 3.6.1, 3.6.2(The Fundamental Theorem of Calculus), and Corollary 3.6.3. TRY to fill in the proof of the **Composition Theorem** if we don't get to it today in class. This is not in the text.

Classwork

1. Page 131: #3.4.5. Show that the "skyscraper" function (graph it), $s(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \leq 1, \end{cases}$ is integrable on $[0, 1]$ and that $\int_0^1 s(x)\,dx = 0$. Use the following process.

a) For $n \in \mathbb{N}$ let P_n be the partition $\{0, \frac{1}{n}, 1\}$. Compute $L(P_n, s)$ and $U(P_n, s)$.

b) Show that s is integrable by showing that for any $\epsilon > 0$, there is a sufficiently large $n \in \mathbb{N}$ such that $U(P_n, s) - L(P_n, s) < \epsilon$.

c) Show that $\int_0^1 s = 0$.

2. Suppose that f is integrable on $[a, b]$.

a) Then by definition f must be bounded on $[a, b]$. In particular, suppose that $m \leq f(x) \leq M$ for all $x \in [a, b]$, where m and M are *any* lower and upper bounds for f on $[a, b]$. Show that $m(b - a) \leq \int_a^b f \leq M(b - a)$.

b) Show: If f is nonnegative and integrable on $[a, b]$, then $0 \leq \int_a^b f$. Use part (a).

The Sup Lemma. Let f be a bounded function on $[a, b]$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Then $M_i - m_i = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}$

Proof. Let $x, y \in [x_{i-1}, x_i]$. By definition of M_i and m_i we have $M_i \geq f(x) \geq m_i$ and $M_i \geq f(y) \geq m_i$. By Problem 1.3.19(d), $M_i - m_i \geq |f(x) - f(y)|$. Since x and y were arbitrary, it now follows that

$$M_i - m_i \geq \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}. \tag{1}$$

Given $\epsilon > 0$. Since $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, then there exist $x \in [x_{i-1}, x_i]$ so that

$$f(x) > M_i - \epsilon/2. \tag{2}$$

Similarly, there exists $y \in [x_{i-1}, x_i]$ so that $f(y) < m_i + \epsilon/2$, which means

$$-f(y) > -m_i - \epsilon/2. \tag{3}$$

ADDING (2) and (3) using Theorem 1.3.8 gives $f(x) - f(y) > M_i - m_i - \epsilon$, and therefore, $|f(x) - f(y)| > M_i - m_i - \epsilon$. It now follows that that

$$\sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} > M_i - m_i - \epsilon.$$

Since this holds for any $\epsilon > 0$, we have (like in presentation problem 1 above)

$$\sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \geq M_i - m_i. \tag{4}$$

The inequalities (1) and (4) imply the desired equality.

Theorem (Composition and Integrability). Suppose that f is integrable on $[a,b]$ and that $c \le f(x) \le d$ for all $x \in [a,b]$.[1] Assume further that g is continuous on $[c,d]$. Then the composite $g \circ f$ is integrable on $[a,b]$.

Proof: Why would this proof be easy if both f and g were continuous? The proof is a bit complicated notationally. We will use Theorem 3.4.9, so let $\epsilon > 0$. (Review the Sup Lemma (Day 26 Handout) before continuing.)

a) Let $K = \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\}$. Why does K exist?

b) Choose $\epsilon' = \frac{b-a+K}{\epsilon} > 0$. (We'll see why later.) g is uniformly continuous on $[c,d]$ by ____________________

c) So there is a $\delta' > 0$ so that whenever $s, t \in [c,d]$ and $|s-t| < \delta'$, then ____________________ $< \epsilon'$. [And for technical reasons, we will want to choose $\delta < \epsilon'$. So let $\delta = \min\{\delta', \epsilon'\}$.]

d) Next, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \delta^2$ by ________________.

e) Now we will show that

$$U(P, g\circ f) - L(P, g\circ f) < \sum_{i=1}^{n} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) < \epsilon.$$

To do this, we separate the set of indices of the partition P into two disjoint sets. On the first set we make $M_i(g\circ f) - m_i(g\circ f)$ small and on the second set we make $\sum(x_i - x_{i-1})$ small. Let

$$A = \{i : M_i(f) - m_i(f) < \delta\} \qquad \text{and} \qquad B = \{i : M_i(f) - m_i(f) \ge \delta\}.$$

If $i \in A$ and $x, y \in [x_{i-1}, x_i]$, then explain why:

$$|f(x) - f(y)| \le M_i(f) - m_i(f) < \delta.$$

f) So if $x, y \in [x_{i-1}, x_i]$, then $|(g\circ f)(x) - (g\circ f)(y)| = |g(f(x)) - g(f(y))| < \epsilon'$ by Step __________.

g) So $M_i(g\circ f) - m_i(g\circ f) \le \epsilon'$ by ____________________________.

h) Adding we get (justify the three inequalities)

$$\sum_{i\in A} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) \le \sum_{i\in A} \epsilon'(x_i - x_{i-1}) \le \sum_{i=1}^{n} \epsilon'(x_i - x_{i-1}) \le \epsilon'(b-a)$$

i) What if $i \in B$? Then $\frac{M_i(f)-m_i(f)}{\delta} \ge 1$ because __________________________. So (justify each inequality)

$$\sum_{i\in B} 1(x_i - x_{i-1}) \le \sum_{i\in B} \left(\frac{M_i(f)-m_i(f)}{\delta}\right)(x_i - x_{i-1}) \le \sum_{\text{all } i} \left(\frac{M_i(f)-m_i(f)}{\delta}\right)(x_i - x_{i-1}) = \frac{U(P,f)-L(P,f)}{\delta} < \delta < \epsilon'.$$

j) By part (a): $M_i(g\circ f) - m_i(g\circ f) \le \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\} = K$ so (justify)

$$\sum_{i\in B} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) \le \sum_{i\in B} K(x_i - x_{i-1}) = K \sum_{i\in B} 1(x_i - x_{i-1}) \le K\epsilon'.$$

k) Now recombine all the indices:

$$\begin{aligned} U(P, g\circ f) - L(P, g\circ f) &= \sum_{i\in A} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) + \sum_{i\in B} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) \\ &\le \epsilon'(b-a) + K\epsilon' = \underline{\hspace{3cm}} < \underline{\hspace{1.5cm}}. \end{aligned}$$

So $g\circ f$ is integrable by ____________________.

[1] The usual notation for $c \le f(x) \le d$ for all $x \in [a,b]$ is to say: $f([a,b]) \subset [c,d]$.

3. (2 Volunteers) Page 131: #3.4.5 (a and b, and probably (c)). Show that the "skyscraper" function (graph it),

$$s(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \leq 1, \end{cases}$$

is integrable on $[0, 1]$ and that $\int_0^1 s(x)\,dx = 0$. Use the following process.

a) For $n \in \mathbb{N}$ let P_n be the partition $\{0, \frac{1}{n}, 1\}$. Compute $L(P_n, s)$ and $U(P_n, s)$.

b) Show that s is integrable by showing that for any $\epsilon > 0$, there is a sufficiently large $n \in \mathbb{N}$ such that $U(P_n, s) - L(P_n, s) < \epsilon$.

c) Show that $\int_0^1 s = 0$.

Proof: Given $\epsilon > 0$. By Problem 1.2.13, there is $n \in \mathbb{N}$ so that $\frac{1}{n} < \epsilon$. Let P_n be the partition $\{0, \frac{1}{n}, 1\}$. Then

$$L(P_n) = 0\left(\tfrac{1}{n} - 0\right) + 0\left(1 - \tfrac{1}{n}\right) = 0$$

and

$$U(P_n) = 1\left(\tfrac{1}{n} - 0\right) + 0\left(1 - \tfrac{1}{n}\right) = \tfrac{1}{n}.$$

Thus, $U(P_n) - L(P_n) = \frac{1}{n} < \epsilon$, so by Theorem 3.4.9, s is integrable on $[0, 1]$.

Now let $P = \{0 = x_0, x_1, \cdots, x_k = 1\}$ be any partition of $[0, 1]$. Then

$$L(P, s) = \sum_{i=1}^{k} m_i(x_i - x_{i-1}) = \sum_{i=1}^{k} 0(x_i - x_{i-1}) = 0.$$

So, since s is integrable,

$$\int_a^b s = \sup_P\{L(P, s)\} = 0.$$

4. (2 Volunteers) Suppose that f is integrable on $[a, b]$.

a) Then by definition f must be bounded on $[a, b]$. In particular, suppose that $m \leq f(x) \leq M$ for all $x \in [a, b]$, where m and M are *any* lower and upper bounds for f on $[a, b]$. Show that $m(b - a) \leq \int_a^b f \leq M(b - a)$.

Proof: Suppose that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Let $P = \{a, b\}$. Let $m_1 = \inf\{f(x) \mid x \in [a, b]\}$ and $M_1 = \sup\{f(x) \mid x \in [a, b]\}$. By definition of infimum and supremum, $m \leq m_1$ and $M_1 \leq M$. Because f is integrable, $L(P, f) \leq \int_a^b f \leq U(P, f)$. Therefore, (by Theorem 1.3.9)

$$m(b - a) \leq m_1(b - a) = L(P, f) \leq \int_a^b f \leq U(P, f) = M_1(b - a) \leq M(b - a).$$

b) Show: If f is nonnegative and integrable on $[a, b]$, then $0 \leq \int_a^b f$. Use part (a).

If f is nonnegative on $[a, b]$, then $0 \leq f(x)$ for all $x \in [a, b]$, i.e., 0 is a lower bound for f. So let $m = 0$ in part (a) to obtain

$$0 = m(b - a) \leq \int_a^b f.$$