Math 331 Homework: Day 30

Read/Review Section 3.6 which we will start on Monday. All of the Theorems in this section are very important. Several should be familiar from Calculus II.

Volunteer to Present:

1. Complete the proof of Part 2 of the Linearity Theorem for Integrals: If f is integrable on $[a, b]$ and c is any constant, then $\int_a^b cf = c\int_a^b f$. Prove the result when $c < 0$. The easiest method is to use previous work (the fact that we know the result is true for $c \geq 0$ and for $c = -1$, but one can also mimic the proof for the case $c > 0$. **Write up the solution which I will copy for the class.**

A great practice problem:

2. f and g are integrable on $[a, b]$, prove fg is integrable on $[a, b]$ using the Square Theorem. Hint: See a problem from Test 2 for a clever way to write fg.

☞ Integrability of Composite Functions

If we do not get to the proof today, there is a powerpoint of the proof on line at our website. You should fill in the blanks on the back of this sheet using the powerpoint.

3.

The Sup Lemma. Let f be a bounded function on $[a, b]$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Then $M_i - m_i = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}$

Proof. Let $x, y \in [x_{i-1}, x_i]$. By definition of M_i and m_i we have $M_i \geq f(x) \geq m_i$ and $M_i \geq f(y) \geq m_i$. By Problem 1.3.19(d), $M_i - m_i \geq |f(x) - f(y)|$. Since x and y were arbitrary, it now follows that

$$M_i - m_i \geq \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}. \tag{1}$$

Given $\epsilon > 0$. Since $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, then there exist $x \in [x_{i-1}, x_i]$ so that

$$f(x) > M_i - \epsilon/2. \tag{2}$$

Similarly, there exists $y \in [x_{i-1}, x_i]$ so that $f(y) < m_i + \epsilon/2$, which means

$$-f(y) > -m_i - \epsilon/2. \tag{3}$$

ADDING (2) and (3) using Theorem 1.3.8 gives $f(x) - f(y) > M_i - m_i - \epsilon$, and therefore, $|f(x) - f(y)| > M_i - m_i - \epsilon$. It now follows that that

$$\sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} > M_i - m_i - \epsilon.$$

Since this holds for any $\epsilon > 0$, we have (like in presentation problem 1 above)

$$\sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \geq M_i - m_i. \tag{4}$$

The inequalities (1) and (4) imply the desired equality.

Theorem (Composition and Integrability). Suppose that f is integrable on $[a,b]$ and that $c \le f(x) \le d$ for all $x \in [a,b]$.[1] Assume further that g is continuous on $[c,d]$. Then the composite $g \circ f$ is integrable on $[a,b]$.

Proof: Why would this proof be easy if both f and g were continuous? The proof is a bit complicated notationally. We will use Theorem 3.4.9, so let $\epsilon > 0$.

a) Let $K = \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\}$. Why does K exist?

b) Choose $\epsilon' = \frac{b-a+K}{\epsilon} > 0$. (We'll see why later.) g is uniformly continuous on $[c,d]$ by ______________________

c) So there is a $\delta' > 0$ so that whenever $s, t \in [c,d]$ and $|s-t| < \delta'$, then ______________________ $< \epsilon'$. [And for technical reasons, we will want to choose $\delta < \epsilon'$. So let $\delta = \min\{\delta', \epsilon'\}$.]

d) Next, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \delta^2$ by ______________.

e) Now we will show that

$$U(P, g\circ f) - L(P, g\circ f) < \sum_{i=1}^{n} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) < \epsilon.$$

To do this, we separate the set of indices of the partition P into two disjoint sets. On the first set we make $M_i(g\circ f) - m_i(g\circ f)$ small and on the second set we make $\sum(x_i - x_{i-1})$ small. Let

$$A = \{i : M_i(f) - m_i(f) < \delta\} \qquad \text{and} \qquad B = \{i : M_i(f) - m_i(f) \ge \delta\}.$$

If $i \in A$ and $x, y \in [x_{i-1}, x_i]$, then explain why:

$$|f(x) - f(y)| \le M_i(f) - m_i(f) < \delta.$$

f) So if $x, y \in [x_{i-1}, x_i]$, then $|(g\circ f)(x) - (g\circ f)(y)| = |g(f(x)) - g(f(y))| < \epsilon'$ by Step __________.

g) So $M_i(g\circ f) - m_i(g\circ f) \le \epsilon'$ by ______________________.

h) Adding we get (justify the three inequalities)

$$\sum_{i\in A} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) \le \sum_{i\in A} \epsilon'(x_i - x_{i-1}) \le \sum_{i=1}^{n} \epsilon'(x_i - x_{i-1}) \le \epsilon'(b-a)$$

i) What if $i \in B$? Then $\frac{M_i(f)-m_i(f)}{\delta} \ge 1$ because ______________________. So (justify each inequality)

$$\sum_{i\in B} 1(x_i - x_{i-1}) \le \sum_{i\in B} \left(\frac{M_i(f) - m_i(f)}{\delta}\right)(x_i - x_{i-1}) \le \sum_{\text{all } i} \left(\frac{M_i(f) - m_i(f)}{\delta}\right)(x_i - x_{i-1}) = \frac{U(P,f) - L(P,f)}{\delta} < \delta < \epsilon'.$$

j) By part (a): $M_i(g\circ f) - m_i(g\circ f) \le \max\{g(t) : t \in [c,d]\} - \min\{g(t) : t \in [c,d]\} = K$ so (justify)

$$\sum_{i\in B} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) \le \sum_{i\in B} K(x_i - x_{i-1}) = K \sum_{i\in B} 1(x_i - x_{i-1}) \le K\epsilon'.$$

k) Now recombine all the indices:

$$\begin{aligned} U(P, g\circ f) - L(P, g\circ f) &= \sum_{i\in A} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) + \sum_{i\in B} [M_i(g\circ f) - m_i(g\circ f)](x_i - x_{i-1}) \\ &\le \epsilon'(b-a) + K\epsilon' = \underline{\qquad\qquad} < \underline{\qquad}. \end{aligned}$$

So $g \circ f$ is integrable by ______________________.

[1] The usual notation for $c \le f(x) \le d$ for all $x \in [a,b]$ is to say: $f([a,b]) \subset [c,d]$.