Math 33 Day 33: Classwork/Practice on limits of sequences

Read Section 4.2 and begin Section 4.3 which we will start next time.

1. Prove or give a counterexample:

a) If $\{s_n\}_{n=1}^{\infty}$ converges to s, then If $\{|s|_n\}_{n=1}^{\infty}$ converges to $|s|$.

b) If $\{|s_n|\}_{n=1}^{\infty}$ converges to $|s|$, then $\{s_n\}_{n=1}^{\infty}$ converges to s.

c) If $\{|s_n|\}_{n=1}^{\infty}$ converges to 0, then $\{s_n\}_{n=1}^{\infty}$ converges to 0.

2. Find an example:

a) a convergent sequence of rational numbers that converges to an irrational number

b) a convergent sequence of irrational numbers that converges to a rational number

3. Given a sequence $\{s_n\}_{n=1}^{\infty}$ and $k \in \mathbb{N}$. Let $\{t_n\}_{n=1}^{\infty}$ be the sequence defined by $t_n = s_{n+k}$. So the terms in the sequences are the same, it's just that $\{t_n\}_{n=1}^{\infty}$ has the first k terms cut off. Prove that $\{t_n\}_{n=1}^{\infty}$ converges if and only if $\{s_n\}_{n=1}^{\infty}$ converges, and if they converge then $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s$. So the convergence of a sequence is not affected by omitting (or changing) a finite number of terms.

4. a) Suppose that $\lim_{n\to\infty} s_n = 0$. If t_n is a bounded sequence prove that $\lim_{n\to\infty} s_n t_n = 0$.

b) Show by an example that the boundedness of $\{t_n\}_{n=1}^{\infty}$ is a necessary condition in part (a).

5. Prove that $\{\ln n\}_{n=1}^{\infty}$ diverges.

6. a) Determine whether $\left\{\frac{\cos n}{n}\right\}_{n=1}^{\infty}$ converges. Prove your result.

b) Determine whether $\left\{\dfrac{5n^2 + \sqrt[3]{n} + \cos n}{2n^2 - 2n + 1}\right\}_{n=1}^{\infty}$ converges.

7. Determine whether $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ converges. Prove your result.

8. For $n \geq 1$, define $a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$.

a) Show that $a_n = 2 - \frac{1}{2^{n+1}}$.

b) Show that $\{a_n\}_{n=1}^{\infty}$ converges to 2.

c) More generally assume $r \neq 1$: For $n \geq 1$, define $a_n = 1 + r + r^2 + \cdots + r^n$.

d) Show that $a_n = \frac{1-r^{n+1}}{1-r}$.

e) For which r will $\{a_n\}_{n=1}^{\infty}$ converge?

9. a) If $\{s_n\}_{n=1}^{\infty}$ converges to s and $s_n \geq 0$ for all $n \in \mathbb{N}$, then $s \geq 0$.

b) If $\{s_n\}_{n=1}^{\infty}$ converges to s and $\{t_n\}_{n=1}^{\infty}$ converges to t and $s_n \geq t_n$ for all $n \in \mathbb{N}$, then $s \geq t$.

c) If $\{s_n\}_{n=1}^{\infty}$ converges to s and $a \leq s_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq s \leq b$.

10. Challenge: Let $k \in \mathbb{R}$. Show that $\displaystyle\lim_{n\to\infty} \frac{k^n}{n!} = 0$. Hint: $\dfrac{k^n}{n!} = \dfrac{k \cdot k \cdot k \cdots k}{1 \cdot 2 \cdot 3 \cdots n}$.