Math 331 Homework: Day 37

Practice and Reading

Review Sections 4.3. Read Section 4.4 which is short.

1. Thinking about series tests:

a) nth term test (for divergence)—often not useful.

b) geometric series test—easy to spot when to use.

c) comparison test—usually compare to a p-series or possibly a geometric series.

d) p-series—easy to spot when to apply; use with comparison test.

e) ratio test—especially useful with factorials and nth powers.

f) root test—useful with nth powers, but ratio test is often a better choice.

Classwork and Practice

1. Try this: Determine whether these series converge or diverge. Justify your answer by showing how the series satisfies the conditions of the test that you apply.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{12k+3}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \ln\left(1+\tfrac{1}{k}\right)^k$ **c)** $\displaystyle\sum_{k=1}^{\infty} \pi^{-k}$ **d)** $\displaystyle\sum_{k=1}^{\infty} k^{-2/3}$ **e)** $\displaystyle\sum_{k=1}^{\infty} \frac{6k+2}{k^2}$

f) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2+k^5}$ **g)** $\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{k}$ **h)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ **i)** $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

2. Determine (justify) whether these series converge or diverge. (Which p in (f) gives convergence?)

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{5^k}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{5^k+3}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \sin(\tfrac{1}{k})$ **d)** $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{2k-1}$ **e)** $\displaystyle\sum_{k=1}^{\infty} \frac{e^{2k}}{k^k}$ **f)** $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$

3. Determine whether these series converge or diverge.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^7+1}}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^k}$ **d)** $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3}$

Hand In Wednesday

I will add a few more problems on Friday. Get started.

1. Prove the Root Test for the case where $r > 1$. Hint: Modify the proof we did for the case $r < 1$. (Use the nth term test?)

2. Determine whether these series converge or diverge. Please carefully justify your answers. Hint for (d): Determine the expression for s_n.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^5+1}}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^3}$ **d)** $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k+1}-\sqrt{k})$

3. a) If $0 < r < 1$, prove that $\sum_{k=1}^{\infty}(k+1)r^k$ converges.

b) I will eventually ask you to prove: If $|r| < 1$, prove that $\sum_{k=1}^{\infty}(k+1)r^k$ converges. (Note: r may be negative.)

4. (Do Problem 4.3.2, but here's a better way to prove it.) Prove the **Limit Comparison Test:** Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are both *positive* series and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

(where L is finite). Then either both series converge or both diverge. *Note:* As with the Comparison test, the Limit comparison test can be modified to require only that a_n and b_n are positive from all n greater than some integer M.

a) Prove that there exists N so that if $n > N$, then $0 < a_n < (L+1)b_n$. Hint: Let $\epsilon = 1$ and first prove $0 < \dfrac{a_n}{b_n} < L + 1$.

b) Use (a) to careful prove: If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. (There should be an argument about tails first.)

c) Prove that $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L} > 0$. Now use this and your work in (b) to prove: If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges.

d) The limit comparison test works well for messy algebraic series by comparing to an appropriate p-series. Use it to determine the convergence or divergence of these series: $\sum_{k=1}^{\infty} \frac{1}{3k^2-4k+5}$ and $\sum_{k=1}^{\infty} \frac{k^2-10}{3k^3-4k+5}$. You should find that when choosing a series for the limit comparison test, you can disregard all but the highest powers of k in the numerator and denominator.

5. Here's a very quick one if you put a couple of ideas together. Prove the **Ratio Test for Sequences**. Assume that $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms and that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$. If $r < 1$, then $\lim_{n\to\infty} a_n = 0$. Hint: Think about the Ratio Test for series.