Math 331 Homework: Day 39

Read Section 4.5 on sequences of functions. Review Section 4.4 and the Lemma and Theorem below.

A Lemma and Theorem about Power Series

The Power Series Lemma. Given the power series $\sum_{k=0}^{\infty} a_k x^k$.

a) If the series converges at some number $x_1 \neq 0$, then it converges absolutely for all $|x| < |x_1|$. Furthermore, the convergence is uniform on any interval of the form $[-s, s]$ where $0 < s < |x_1|$.

b) If the series diverges at some number x_2, then it diverges for all $x > |x_2|$.

Proof: **(a)** Since $\sum_{n=0}^{\infty} a_n x_1^n$ converges, $\lim_{n\to\infty} a_n x_1^n = 0$. In particular, there is an integer N such that for all $n > N$, we have $|a_n x_1^n| < 1$.

Choose any positive number x such that $0 < |x| < |x_1|$. Let $r = \left|\frac{x}{x_1}\right|$, so $0 < r < 1$. If $n > N$, then

$$|a_n x^n| = |a_n x_1^n| \left|\frac{x}{x_1}\right|^n \leq \left|\frac{x}{x_1}\right|^n \leq r^n.$$

But the geometric series $\sum_{n=N+1}^{\infty} r^n$ is convergent, so by comparison $\sum_{n=N+1}^{\infty} |a_n x^n|$ is absolutely convergent, hence the entire series $\sum_{k=0}^{\infty} a_k x^k$ converges (absolutely).

To prove **(b)**, take any y with $|y| > |x_2|$. If the power series converged at y, by **(a)** the series would converge absolutely for all x with $|x| < |y|$, including $x = x_2$, which contradicts the hypothesis. ∎

The Power Series Theorem. Let $\sum_{k=0}^{\infty} a_k x^k$ be a power series. Exactly one of the following holds:

1. $\sum_{k=0}^{\infty} a_k x^k$ converges only at 0.

2. $\sum_{k=0}^{\infty} a_k x^k$ converges for all x.

3. There exists a positive real number r such that $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x| < r$ and diverges if $|x| > r$. (It may or may not converge at $x = r$ or $-r$.)

Note: r is called the **radius of convergence**. $r = 0$ in (1) and $r = \infty$ in (2).

Proof: Assume neither (1) nor (2) hold. [We must show (3) holds.] Since neither (1) nor (2) hold, then $\sum_{k=0}^{\infty} a_k x^k$ converges at some $x_1 \neq 0$ and diverges at some $x_2 \neq 0$. We will use the least upper bound axiom to determine the radius of convergence r. Let

$$S = \left\{ x \in \mathbb{R} : \sum_{k=0}^{\infty} a_k x^k \text{ converges} \right\}.$$

S is not the empty set since it contains ____________. By the Power Series Lemma (b), S is _______________________ by $|x_2|$. So S has a least upper bound r. Using the lemma again, the power series converges for all x with $|x| < |x_1|$, so $r \geq |x_1| > 0$.

Now we show that the power series converges for any x such that $|x| < r$. If $|x| < r$, since r is the least upper bound of S there is some number y in S such that $|x| < y \leq r$. By the Power Series Lemma (a), if the series converges at y, then _______________________ .

Finally, observe that if the series converged at some x with $|x| > r$, we could choose z such that $r < z < |x|$. By the Power Series Lemma (b), the series would now _______________ at z, which contradicts the fact that r is _______________________ of S. Consequently, $\sum_{k=0}^{\infty} a_k x^k$ diverges for all x such that $|x| > r$. ∎

Classwork

4. Classify each of these series as conditionally or absolutely convergent or as divergent. Please carefully justify your answers.

a) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1}$ **b)** $\displaystyle\sum_{k=0}^{\infty} (-1)^{k} \frac{1}{k!}$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-k)^k}{k!}$ **d)** $\displaystyle\sum_{k=1}^{\infty} \pi^{-2k} \sin k$

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5. Find the radius of convergence R for each and then the corresponding interval of convergence (endpoints). Please carefully justify your answers.

a) $\displaystyle\sum_{k=1}^{\infty} 2kx^k$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{3}\right)^k$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{k!x^k}{k^k}$ **d)** $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!}$ **e)** $\displaystyle\sum_{k=1}^{\infty} k^k x^k$