Math 331 Homework: Day 40

Finish reading Section 4.5. Review Section 4.4.

Problem from a previous final

1. This problem was on the final exam in 2009. It is named after a student in the course that year who asked this question. Recall the Two Point Lemma whose statement I have modified just slightly to make your life easier: *If* $\lim_{x\to a} f(x)$ *exists, then for every* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $0 < |x-a| < \delta/2$ *and* $0 < |y-a| < \delta/2$, *then* $|f(x) - f(y)| < \epsilon/2$. The converse is true, but is difficult to prove without a knowledge of sequences, which we now have. The problem is that to show that $\lim_{x\to a} f(x)$ exists using the limit definition, one needs to know what the limit L **is** and we don't have that information. Here's the the converse: (**Charlie's Theorem**): *Assume that for every* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $0 < |x-a| < \delta/2$ *and* $0 < |y-a| < \delta/2$, *then* $|f(x) - f(y)| < \epsilon/2$. *Then* $\lim_{x\to a} f(x)$ *exists.*

a) Given $\epsilon > 0$. There exists $N_1 \in \mathbb{N}$ so that $\frac{1}{N_1} < \delta/2$ by ______________________________.

b) Define the sequence $\{b_n\}_{n=1}^{\infty}$ by $b_n = f(a + \frac{1}{n})$. Prove that $\{b\}_{n=1}^{\infty}$ is a Cauchy sequence. Hint: Use N_1.

c) $\{b\}_{n=1}^{\infty}$ has a limit B because ______________________________. We will show $\lim_{x\to a} f(x) = B$.

d) There exists $N_2 \in \mathbb{N}$ so that if $n > N_2$, then $|b_n - B| < \epsilon/2$ by ______________________________.

e) Set $N = \max\{N_1, N_2\}$. Now let x be any point so that $0 < |x-a| < \delta/2$. Prove that $|f(x) - B| < \epsilon$. Hint: $|f(x) - B| = |f(x) - b_{N+1} + b_{N+1} - B|$. Use your previous work, the hypotheses, and remember what b_{N+1} means.

Hand In Tuesday

You can have until Wednesday if you ask.

1. Classify each of these series as conditionally or absolutely convergent or as divergent. Carefully justify your answers. (Rewrite and simplify (d) first.)

a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}k}{k^2 + \frac{1}{2}}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{\sqrt{k^6+1}}$ **c)** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \left(\frac{k}{k+1}\right)^k$ **d)** $\displaystyle\sum_{k=1}^{\infty} \cos(\pi k)\left(\frac{\pi}{e}\right)^{-2k}$

Maple commands: You may want to use Maple to look at some of the following sequences. (In the Math Lab, there are guest accounts (temp1, temp2, temp3, temp4, with passwords godel, escher, bach, knuth.) The first command loads the plotting macros. The second will produce an animation. The third displays all the terms at once. Change f_n, the interval, and the parameters for n as needed.

```
with(plots):
display(seq(plot(sin(n*x)/n, x = 0 .. 2*Pi), n = 1 .. 50), insequence = true);
display(seq(plot(sin(n*x)/n, x = 0 .. 2*Pi), n = 1 .. 50));
```

2. Page 204 #4.5.2. You should justify your limit calculations (see Problem 4.2.4). You may wish to look at `http://math.hws.edu/FoundationsOfAnalysis/` Use the pull-down menu to select the correct example. Or use Maple.

3. Page 204 #4.5.3. You may use your Calculus Knowledge to determine the necessary limits. You may wish to look at the animation applet at `http://math.hws.edu/FoundationsOfAnalysis/` Or use Maple.

4. **a)** Find the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$, where $f_n(x) = \frac{n}{x+n}$ on $[0, \infty)$.

b) Show that this sequence converges uniformly on $[0, 10]$.

5. For $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$, define $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$.

a) Print a plot of $f_n(x)$ that shows the first say the first 40 or so terms. Modify the Maple commands above. You will also want to see the animation, too.

b) Use the definition of uniform convergence to prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a limit function f. (You should be able to tell what the limit function is.)

c) What is $f'(x)$. Show that $f'(x) \neq \lim_{n\to\infty} f_n'(x)$ even though the convergence above was uniform. Hint: Plot $\{f_n'\}_{n=1}^{\infty}$. It suffices to show that $f'(x) \neq \lim_{n\to\infty} f_n'(x)$ at some point in the interval. Which one is convenient?

d) Explain why this does not contradict Theorem 4.5.4.

6. Find the radius of convergence R and the corresponding interval of convergence (check endpoints) for each of these series. Please carefully justify your answers.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k x^k}{k}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \left(\frac{x}{k}\right)^k$ **c)** $\displaystyle\sum_{k=1}^{\infty} \frac{3^{2k} x^k}{k}$

7. Page 205 4.5.7(a). Hint: Our old buddy, the triangle inequality. Extra Credit: Do part (b).

8. Extra Credit: Page 205 4.5.8.