The Field Axioms

Axioms for Addition:

1. Closure: For all $a, b \in \mathbb{F}$, there is a unique element $a + b \in \mathbb{F}$.
2. Commutativity: For all $a, b \in \mathbb{F}$, $a + b = b + a$.
3. Associativity: For all $a, b, c \in \mathbb{F}$, $a + (b + c) = (a + b) + c$.
4. Identity: There exists an element of $\mathbb{F}$ denoted by 0 so that for all $a \in \mathbb{F}$, $a + 0 = a$.
5. Inverses: For every $a \in \mathbb{F}$, there exists an element $-a \in \mathbb{F}$ so that $a + (-a) = 0$.

Axioms for Multiplication:

6. Closure: For all $a, b \in \mathbb{F}$, there is a unique number $a \cdot b$ (often written ab) in $\mathbb{F}$.
7. Commutativity: For all $a, b \in \mathbb{F}$, $a \cdot b = b \cdot a$.
8. Associativity: For all $a, b, c \in \mathbb{F}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
9. Identity: There exists an element of $\mathbb{F}$ denoted by 1 which does not equal 0, such that for every $a \in \mathbb{F}$, $a \cdot 1 = a$.
10. Inverses: For every $a \in \mathbb{F}$ with $a \neq 0$, there exists an element $a^{-1} \in \mathbb{F}$ so that $a \cdot (a^{-1}) = 1$.

Distributive Axiom:

11. For all $a, b \in \mathbb{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Basic Facts

THEOREM 1.0.1. Let $\mathbb{F}$ be a field.

(1) If $a \in \mathbb{F}$, then $a \cdot 0 = 0$.

(2) 0 does not have a multiplicative inverse.

THEOREM 1.0.2 (Uniqueness of Identities and Inverses). The additive and multiplicative identities are unique. Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$.

(a) If $a + b = a$, then $b = 0$;

(b) If $a \cdot b = a$ and $a \neq 0$, then $b = 1$.

For each element the additive and multiplicative inverses are unique. That is:

(c) If $a + b = 0$, then $b = -a$.

(d) If $a \cdot b = 1$, then $b = a^{-1}$.

COROLLARY 1.0.3. Let $\mathbb{F}$ be a field.

(1) $-0 = 0$ and $1^{-1} = 1$.

(2) If $a \in \mathbb{F}$, then $-(-a) = a$.

(3) If $a \in \mathbb{F}$ and $a \neq 0$, then $(a^{-1})^{-1} = a$.

THEOREM 1.0.4. Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$. If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

THEOREM 1.0.5. Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$. Then

(a) $(-a) \cdot b = -(a \cdot b)$ and, in particular, $(-1) \cdot b = -b$;

(b) $(-a) \cdot (-b) = a \cdot b$.