Reading, Practice

Review Section 3.3 carefully. We are done with differential calculus! Read ahead in Section 3.4. We will begin integration next time.

The Mean Value Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Using the MVT

By the end of class you should be able to prove all of the Results 1 through 5 below.

1. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Under these hypotheses: Let $c, d \in [a, b]$ with $c < d$. Use the MVT on the interval $[c, d]$ and the given information to determine the relationship between $f(c)$ and $f(d)$ and prove the resut.

(*a*) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

(*b*) If $f'(x) > 0$ for all $x \in (a, b)$ and if $c, d \in [a, b]$ with $d < d$, then $f(c) < f(d)$ (that is, f is **increasing** on $[a, b]$.

(*c*) If $f'(x) < 0$ for all $x \in (a, b)$ and if $x, y \in (a, b)$ with $x < y$, then $f(x) > f(y)$ (that is, f is **decreasing** on $[a, b]$.

(*d*) If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is one-to-one on $[a, b]$.

1. (*a*) Jack, Kyle
(*b*) Lillie, Alana
(*c*) Nan, Weixiang
(*d*) Liv, Michael, David

2. Suppose that f, g are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) = f'(x)$ for all $x \in (a, b)$. Then there is a constant k so that $g(x) = f(x) + k$ on $[a, b]$. Hint: Consider $h(x) = g(x) - f(x)$

2. Jack, Kyle

☞ The next three problems all use the same idea: Apply the MVT to the correct function $f(t)$ on the interval $[a, x]$, where a is a constant that depends on the question.

3. Use the MVT to prove: If $x \geq 0$, then $\sin x \leq x$. (Assume Calculus I knowledge.) Hint: The result is clearly true if $x = 0$ (right?). So assume $x > 0$. Let $f(t) = \sin t$ on $[0, x]$.

4. Use the MVT to prove **Bernoulli's Inequality**: For all $x > 0$ and for all $n \in \mathbb{N}$,

$$(1 + x)^n > 1 + nx.$$

What's $f(t)$ this time? Note: This can be done by induction, but it is quicker with the MVT.

3. Lillie, Alana
4. Nan, Weixiang
5. Liv, Michael, David

5. Prove: If $x > 1$, then $\frac{x-1}{x} < \ln x < x - 1$. What's $f(t)$ this time?

6. **The Cauchy Mean Value Theorem.** Suppose that f and g are continuous on the closed, bounded interval $[a, b]$ and are differentiable on (a, b). Then there is a point c strictly between a and b such that

$$\big(g(b) - g(a)\big)f'(c) = \big(f(b) - f(a)\big)g'(c).$$

(*a*) Define the auxiliary function $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$. Show that h is continuous on $[a, b]$ and differentiable on (a, b).

(*b*) Show that $h(a) = h(b)$.

(*c*) Apply the MVT to h and show that c is the desired point.

6. Jack, Kyle

7. Problem 7 on the back, assuming the IVTD which you are proving for Homework.

7. Everyone else

Over

Comments on the Current Assignment

Differentiability on Closed Intervals. We say g is **differentiable on the closed interval** $[a, b]$ if g is differentiable at each point in the open interval (a, b) and the appropriate one-sided derivatives exist at a and b. Specifically

1. g is differentiable at each $x \in (a, b)$,
2. $\lim_{x \to a^+} \dfrac{g(x) - g(a)}{x - a}$ exists (and is denoted by $g'(a)$), and $\lim_{x \to b^-} \dfrac{g(x) - g(b)}{x - b}$ exists (and is denoted by $g'(b)$).

Note: All basic derivative rules (e.g., sum, product) carry over to functions differentiable on closed intervals.

Current Homework

5. Intermediate Value Theorem for Derivatives. If f is differentiable on $[a, b]$ and $f'(a) < k < f'(b)$, then there is a $c \in (a, b)$ with $f'(c) = k$. A similar result holds if $f'(a) > k > f'(b)$. (Note: We cannot apply the IVT because we do not know that f' is continuous on $[a, b]$.)

(*a*) Consider the auxiliary function $g(x) = f(x) - kx$, for $x \in [a, b]$. Since f and x are differentiable on $[a, b]$ it follows that g is differentiable on $[a, b]$. Show that $g'(a) < 0 < g'(b)$.

(*b*) Prove that g has a minimum point $c \in [a, b]$.

(*c*) From part (a), $0 < g'(b) = \lim_{x \to b^-} \dfrac{g(x) - g(b)}{x - b}$. Use the definition of one-sided limit to prove that there exists $\delta > 0$ so that if $-\delta < x - b < 0$, then $0 < \frac{g(x)-g(b)}{x-b}$. Hint: Let $\varepsilon = g'(b)$.

(*d*) With this same δ prove: If $-\delta < x - b < 0$, then $g(x) < g(b)$. [This shows that $g(b)$ is NOT the minimum value of g. A similar argument shows that $g(a)$ is also not the minimum value of g. In other words, $c \neq a$ and $c \neq b$.]

(*e*) So $c \in (a, b)$. Prove $g'(c) = 0$ and then show $f'(c) = k$.

6. True or False: The Dirichlet function $D(x)$ is the derivative of some function $F(x)$ on the interval $[a, b]$. (Is $D(x) = F'(x)$ for some function F?) Explain.

On the next assignment–or not

7. Corollary of IVTFD. If f is differentiable on $[a, b]$ and $f'(x) \neq 0$ for all $x \in (a, b)$, then either $f'(x) < 0$ for all $x \in [a, b]$ or $f'(x) > 0$ for all $x \in [a, b]$. (So from Problem 1, f is either always increasing or always decreasing on $[a, b]$.)

Solutions to In-Class Problems

1. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Under these hypotheses: Let $c,d \in [a,b]$ with $c < d$. Use the MVT on the interval $[c,d]$ and the given information to determine the relationship between $f(c)$ and $f(d)$ and prove the resut.

(*a*) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

(*b*) If $f'(x) > 0$ for all $x \in (a,b)$ and if $c,d \in [a,b]$ with $d < d$, then $f(c) < f(d)$ (that is, f is **increasing** on $[a,b]$.

(*c*) If $f'(x) < 0$ for all $x \in (a,b)$ and if $x,y \in (a,b)$ with $x < y$, then $f(x) > f(y)$ (that is, f is **decreasing** on $[a,b]$.

(*d*) If $f'(x) \neq 0$ for all $x \in (a,b)$, then f is one-to-one on $[a,b]$. [Hint: Contradiction or contraposition is easy.]

1. (*a*) Jack, Kyle
(*b*) Lillie, Alana
(*c*) Nan, Weixiang
(*d*) Liv, Michael, David

PROOF (a). Let $c,d \in [a,b]$ with $c < d$. It suffices to show that $f(c) = f(d)$. Since f is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) = 0$ for all $x \in (a,b)$, then f is continuous on $[c,d]$ and differentiable on (c,d). So the MVT applies to f on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$f'(z) = \frac{f(d) - f(c)}{d - c}.$$

Since $f'(z) = 0$, it follows that $f(d) = f(c)$ for all $c,d \in [a,b]$. That is, f is constant.

PROOF (b). Let $c,d \in [a,b]$ with $c < d$. Show that $f(c) < f(d)$. Since f is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) > 0$ for all $x \in (a,b)$, then f is continuous on $[c,d]$ and differentiable on (c,d). So the MVT applies to f on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$\frac{f(d) - f(c)}{d - c} = f'(z) > 0.$$

Because $c < d$, it follows$f(d) - f(c) > 0$. That is, $f(c) < f(d)$.

PROOF (c). Let $c,d \in [a,b]$ with $c < d$. Show that $f(c) > f(d)$. Since f is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) > 0$ for all $x \in (a,b)$, then f is continuous on $[c,d]$ and differentiable on (c,d). So the MVT applies to f on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$\frac{f(d) - f(c)}{d - c} = f'(z) < 0.$$

Because $c < d$, Ii follows$f(d) - f(c) < 0$. That is, $f(c) > f(d)$.

PROOF (d: Contraposition.). Assume f is not one-to-one on $[a,b]$. Then there exist $c,d \in [a,b]$ with $c < d$ such that $f(c) = f(d)$. Since f is continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) \neq 0$ for all $x \in (a,b)$, then f is continuous on $[c,d]$ and differentiable on (c,d). So the MVT applies to f on $[c,d]$. Therefore, there is a point $z \in [c,d]$ so that

$$f'(z) = \frac{f(d) - f(c)}{d - c} = \frac{0}{d - c} = 0.$$

2. Suppose that f, g are continuous on $[a,b]$ and differentiable on (a,b) with $g'(x) = f'(x)$ for all $x \in (a,b)$. Then there is a constant k so that $g(x) = f(x) + k$ on $[a,b]$. Hint: Consider $h(x) = g(x) - f(x)$

2. Jack, Kyle

PROOF. Let $h(x) = g(x) - f(x)$ for all $x \in [a,b]$. Since f, g are continuous on $[a,b]$ and differentiable on (a,b), then so is h. But $h'(x) = g'(x) - f'(x) = 0$ for all $x \in (a,b)$. So by Problem 1(a), $h(x) = k$ is $g(x) - f(x) = k$ and so $g(x) = f(x) + k$.

☞ The next three problems all use the same idea: Apply the MVT to the correct function $f(t)$ on the interval $[a,x]$, where a is a constant that depends on the question.

3. Use the MVT to prove: If $x \geq 0$, then $\sin x \leq x$. (Assume Calculus I knowledge.) Hint: The result is clearly true if $x = 0$ (right?). So assume $x > 0$. Let $f(t) = \sin t$ on $[0, x]$.

3. Lillie, Alana

4. Nan, Weixiang

5. Liv, Michael, David

PROOF. The result is clearly true if $x = 0$ because $sin(0) = 0$. So assume $x > 0$. Let $f(t) = \sin t$ on $[0, x]$ where it is continuous and f is differentiable on $(0, x)$. By the MVT, there is a point $c \in [0, x]$ so that

$$1 \geq \cos(c) = f'(c) = \frac{\sin(x) - \sin(0)}{x - 0} = \frac{\sin(x)}{x}.$$

Consequently, $x \geq \sin(x)$.

4. Use the MVT to prove **Bernoulli's Inequality**: For all $x > 0$ and for all $n \in \mathbb{N}$,

$$(1 + x)^n > 1 + nx.$$

What's $f(t)$ this time? Note: This can be done by induction, but it is quicker with the MVT.

PROOF. Let $x > 0$ and $n \in \mathbb{N}$. Let $f(t) = (1 + t)^n$ on $[0, x]$. Since f is a polynomial, it is continuous and differentiable everywhere. By the MVT, there is a point $c \in [0, x]$ so that

$$n(1 + t)^{n-1} = f'(c) = \frac{(1 + x)^n - (1 + 0)^n}{x - 0}.$$

Consequently,

$$nx(1 + t)^{n-1} = (1 + x)^n - 1.$$

Since $t > 0$, it follows that $(1 + t) > 1$, so $(1 + t)^{n-1} > 1^{n-1} = 1$. Therefore,

$$nx < nx(1 + t)^{n-1} = (1 + x)^n - 1 \quad \text{or} \quad 1 + nx < (1 + x)^n.$$

5. Prove: If $x > 1$, then $\frac{x - 1}{x} < \ln x < x - 1$. What's $f(t)$ this time?

PROOF. Let $f(t) = \ln t$ on $[1, x]$. From Calc 1, f is continuous and differentiable on $[1, x]$. By the MVT, there is a point $c \in [1, x]$ so that

$$\frac{1}{c} = f'(c) = \frac{\ln x - \ln 1}{x - 1} = \frac{\ln x}{x - 1}.$$

Consequently,

$$\frac{x - 1}{c} = \ln x.$$

Since $1 < c < x$, it follows that

$$\frac{x - 1}{x} < \frac{x - 1}{c} = \ln x < \frac{x - 1}{1} = x - 1.$$

6. **The Cauchy Mean Value Theorem.** Suppose that f and g are continuous on the closed, bounded interval $[a, b]$ and are differentiable on (a, b). Then there is a point c strictly between a and b such that

6. Jack, Kyle

$$\big(g(b) - g(a)\big)f'(c) = \big(f(b) - f(a)\big)g'(c).$$

PROOF. Define the auxiliary function $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$. Since f, g are continuous on $[a, b]$ and differentiable on (a, b), so are the constant multiples $(g(b) - g(a))f(x)$ and $(f(b) - f(a))g(x)$ and consequently, so is their difference $h(x)$. Next

$$h(a) = (g(b) - g(a))f(a) - (f(b) - f(a))g(a) = g(b)f(a) - f(b)g(a) = (g(b) - g(a))f(b) - (f(b) - f(a))g(b)$$

so $h(a) = h(b)$. Applying the MVT to h, there is point $c \in [a, b]$ so that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{0}{b - a} = 0.$$

But then

$$h'(c) = (g(b) - g(a))f'(c) - (f(b) - f(a))g'(c) = 0$$

and the result follows.

7. **Corollary of IVTFD.** If f is differentiable on $[a, b]$ and $f'(x) \neq 0$ for all $x \in (a, b)$, then either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$. (So from Problem 1, f is either always increasing or always decreasing on $[a, b]$.)

PROOF (Contradiction). Suppose that there exist $c, d \in [a, b]$ with $f'(c) < 0$ and $f'(d) > 0$. Since f is differentiable (and so continuous) on $[a, b]$, then f is differentiable and continuous on $[c, d]$, so the IVTFD applies there. Since $f'(c) < 0 < f'(d)$, there is point $z \in [c, d]$ so that $f'(z) = 0$. Since $[c, d] \subseteq [a, b]$, it follows that $z \in [a, b]$. This contradicts that $f'(x) \neq 0$ for all $x \in (a, b)$.