# *Dedekind Cuts and Real Numbers*

**DEFINITION 1.2.1.** A *Dedekind cut* is a subset $\alpha$ of the rational numbers $\mathbb{Q}$ with the following properties:

1. $\alpha$ is not empty and $\alpha \neq \mathbb{Q}$;
2. if $p \in \alpha$ and $q < p$, then $q \in \alpha$;
3. if $p \in \alpha$, then there is some $r \in \alpha$ such that $r > p$ (i.e., $\alpha$ has no maximal element).

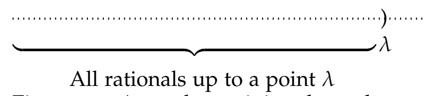


Figure 1.1: A cut determining the real number $\lambda$.

**DEFINITION 1.2.2.** The set of *real numbers* $\mathbb{R}$ is the collection of all Dedekind cuts. Two real numbers $\alpha$ and $\beta$ are *equal* if and only if both cuts are the same subset of $\mathbb{Q}$.

> **NOTATION.** ☞ In the material that follows, unaccented lowercase letters of the Roman alphabet $(a, b, c, \ldots, p, q, r, s, t, \ldots, z)$ will always indicate **rational numbers**, while lower case Greek letters $(\alpha, \beta, \gamma, \lambda, \ldots)$ will indicate **Dedekind cuts** (real numbers).

**FACT 1.1.** If $\alpha$ and $\beta$ are cuts and $\alpha \neq \beta$, then either $\alpha \subset \beta$ or $\beta \subset \alpha$ (but not both).

ANALYSIS: This property is not true for sets in general. For example, if $A = \{x, y, z\}$ and $B = \{w, x\}$, then $A \neq B$, $A \not\subset B$, and $B \not\subset A$. ♢

STRATEGY: Proving $P \Rightarrow (Q \vee R)$ is equivalent to proving $[P \wedge (\sim Q)] \Rightarrow R$. ♢

*Proof.* Assume $\alpha \neq \beta$ and $\alpha \not\subset \beta$. We must show $\beta \subset \alpha$. Let $b \in \beta$. (Show $b \in \alpha$). Since $\alpha \not\subset \beta$, $\exists a \in \alpha$ so that $a \notin \beta$. Thus, $a \neq b$. By cut property 2 for $\beta$, it follows that $a \not< b$. So $b < a$. By cut property 2 for $\alpha$, it follows that $b \in \alpha$. □

**EXERCISE 1.2.3.** Let $\alpha$ be a cut. If $c \in \mathbb{Q}$ and $c \notin \alpha$, then $c > p$ for all $p \in \alpha$. (What method of proof is helpful?)

**EXERCISE 1.2.4 (Corollary to Exercise 1.2.3).** Let $\alpha$ be a cut and $c, d \in \mathbb{Q}$. If $d > c$ and and $c \notin \alpha$, then $d \notin \alpha$.

**THEOREM 1.2.5.** For any rational number $r$ the set $\hat{r} = \{q \in \mathbb{Q} : q < r\}$ is a cut, and hence a real number.

*Proof.* This is a homework problem. Demonstrate that $\hat{r}$ satisfies the three conditions of a cut.

1. Prove $\hat{r} \neq \emptyset$. (Find a rational in $\hat{r}$.) Also prove $\hat{r} \neq \mathbb{Q}$. (Find a rational not in $\hat{r}$.)
2. Let $p \in \hat{r}$ and $q < p$. Show $q \in \hat{r}$.
3. Let $p \in \hat{r}$. Find a rational $q$ so that $q > p$ and $q \in \hat{r}$.

□

**EXAMPLE 1.2.6.** Define the cuts $\hat{1}$, $\widehat{-\frac{2}{3}}$, and $\hat{0}$.

**EXERCISE 1.2.7.** In trying to define the cut for the real number $\sqrt{2}$,

1. Jon suggests: $\sqrt{2} = \left\{q \in \mathbb{Q} : q < \sqrt{2}\right\}$. Does this work?
2. Jane suggests: $\sqrt{2} = \{q \in \mathbb{Q} : q^2 < 2\}$. Does this work?
3. Jim suggests: Does $\sqrt{2} = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}$ work?

**EXAMPLE 1.2.8.** What cut (real number) does the following represent:

$$\left\{q \in \mathbb{Q} : \exists n \in \mathbb{N} \text{ such that } q \leq \left(1 + \frac{1}{n}\right)^n\right\}?$$

**DEFINITION 1.2.9.** For two real numbers (cuts) $\alpha$ and $\beta$, we say $\alpha < \beta$ if $\alpha \subset \beta$. (The inclusion is proper, $\alpha \neq \beta$.)

**EXERCISE 1.2.10.** Let $r \in \mathbb{Q}$. If $\hat{r} < \alpha$, then $r \in \alpha$. What method of proof might be useful?

**THEOREM 1.2.11** (Trichotomy). For any real number (cut) $\alpha$, exactly one of the following holds:

$$\alpha > \hat{0}, \quad \alpha = \hat{0}, \quad \text{or} \quad \alpha < \hat{0}.$$

*Proof.* Assume $\alpha$ is a cut (real). Either $\alpha = \hat{0}$ or $\alpha \neq \hat{0}$. In the first case, if $\alpha = \hat{0}$, then $\alpha \not\subset \hat{0}$ and $\hat{0} \not\subset \alpha$, so $\alpha \not< \hat{0}$ and $\hat{0} \not< \alpha$.

In the other case, $\alpha \neq \hat{0}$, so by Fact 1.1 either $\alpha \subset \hat{0}$ or $\hat{0} \subset \alpha$ (but not both). That is, if $\alpha \neq \hat{0}$, either $\alpha < \hat{0}$ or $\alpha > \hat{0}$ (but not both). □

**DEFINITION 1.2.12.** Let $\alpha, \beta$ be a cut. We say $\alpha$ is **positive** if $\alpha > \hat{0}$ and $\alpha$ is **negative** if $\alpha < \hat{0}$.

**EXERCISE 1.2.13.** True or false (explain):

1. A cut $\alpha$ is positive if and only if $0 \in \alpha$.
2. A cut $\alpha$ is negative if and only if $0 \notin \alpha$.

## *Addition*

**DEFINITION 1.2.14.** Let $\alpha, \beta \in \mathbb{R}$ (be cuts). Then the **sum** of $\alpha$ and $\beta$ is the set

$$\alpha + \beta = \{r \in \mathbb{Q} : r < p + q, \text{ where } p \in \alpha \text{ and } q \in \beta\}.$$

**EXERCISE 1.2.15.** Let $\alpha, \beta \in \mathbb{R}$ (be cuts). If $a \in \alpha$ and $b \in \beta$, then $a + b \in \alpha + \beta$.

**THEOREM 1.2.16.** If $\alpha, \beta \in \mathbb{R}$, then $\alpha + \beta \in \mathbb{R}$, i.e., $\alpha + \beta$ is a cut.

*Proof.* We must show that $\alpha + \beta$ satisfies the three cut properties.

1. (a) Show $\alpha + \beta \neq \emptyset$ and (b) $\alpha + \beta \neq \mathbb{Q}$.
2. Let $x \in \alpha + \beta$ and let $y < x$, where $x, y \in \mathbb{Q}$. Show $y \in \alpha + \beta$.
3. Let $x \in \alpha + \beta$. Show there exists $z \in \alpha + \beta$ with $z > x$.

□

Exercise 1.2.15: Mike
1(a): David and Liv
1(b): Jack and Kyle
2: Alana and Lillie
3: Nan, Weixiang

**COROLLARY 1.2.17.** Let $\alpha$ and $\beta$ be reals (cuts). Define $\alpha \oplus \beta = \{p + q : p \in \alpha, q \in \beta\}$. Then

$$\alpha + \beta = \alpha \oplus \beta.$$

What is the difference in the definitions of $\alpha + \beta$ and $\alpha \oplus \beta$?

ANALYSIS: Remember, cuts (real numbers) are sets of rationals. This is a set equality so it requires two subset proofs. ♢

*Proof.* Check that $\alpha \oplus \beta \subseteq \alpha + \beta$ is an immediate consequence of Exercise 1.2.15.

Now we show $\alpha + \beta \subseteq \alpha \oplus \beta$. Let $r \in \alpha + \beta$. (Show $r \in \alpha \oplus \beta$.) By definition of $\alpha + \beta$, we have $r < p + q_1$ where $p \in \alpha$ and $q_1 \in \beta$. Let $q = r - p$. Then $q \in \mathbb{Q}$ because $r, p \in \mathbb{Q}$. Further $r < p + q_1$ so $q = r - p < q_1$. So by cut property 2 for $\beta$, it follows that $q \in \beta$. Since $q = r - p$, it follows that $r = p + q$ where $p \in \alpha$ and $q \in \beta$. This means $r \in \alpha \oplus \beta$. □

Using Corollary 1.2.17 and properties of the rationals, it is easy to show

**EXERCISE 1.2.18.** Addition of real numbers is commutative and associative. That is,

1. For all reals (cuts) $\alpha$ and $\beta$, we have $\alpha + \beta = \beta + \alpha$.
2. For all reals (cuts) $\alpha$, $\beta$, and $\gamma$, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

**THEOREM 1.2.19.** For any real number $\alpha$, $\alpha + \hat{0} = \alpha$.

STRATEGY: Remember, cuts (real numbers) are sets of rationals. This is a set equality so it requires two subset proofs. Make use of Corollary 1.2.17. ♢

*Proof.* First we show $\alpha + \hat{0} \subseteq \alpha$. Let $r \in \alpha + \hat{0}$. (Show $r \in \alpha$.) By Corollary 1.2.17 $r = p + q$, where $p \in \alpha$ and $q \in \hat{0}$. By definition of $\hat{0}$, we have $q < 0$. Thus $r = p + q < p + 0 = p$, so by cut property (2) of $\alpha$, it follows that $r \in \alpha$.

Next we show $\alpha \subseteq \alpha + \hat{0}$. Let $p \in \alpha$. (Show $p \in \alpha + \hat{0}$.) Since $\alpha$ is a cut, by property (3) there exists a rational $s \in \alpha$ so that $p < s$. Then $p - s < 0$ and $p - s \in \mathbb{Q}$ because $s, p \in \mathbb{Q}$. So by definition $p - s \in \hat{0}$. Then $p = s + (p - s)$, where $s \in \alpha$ and $p - s \in \hat{0}$. So by Corollary 1.2.17 $p \in \alpha + \hat{0}$. □

**DEFINITION 1.2.20.** We say that $\hat{0}$ is the **additive identity** of $\mathbb{R}$.

**THEOREM 1.2.21.** The **additive identity** of $\mathbb{R}$ is unique. That is, if $\lambda \in \mathbb{R}$ (is a cut) and and $\alpha + \lambda = \alpha$ for all $\alpha \in \mathbb{R}$, then $\lambda = \hat{0}$.

**EXERCISE 1.2.22.** Prove this result. You should have done uniqueness proofs of identity elements in Math 135 and 204.

**DEFINITION 1.2.23.** If for two real numbers $\alpha$ and $\beta$ we have $\alpha + \beta = \hat{0}$, then we say that $\beta$ is the **additive inverse** of $\alpha$.

**THEOREM 1.2.24.** Any real number $\alpha$ has an additive inverse.

ANALYSIS: ☞ This is the hardest proof in this section and we will not cover it in class. But read through it. ♢

*Proof.* Let $\alpha \in \mathbb{R}$. Let

$$\beta = \{q \in \mathbb{Q} : \exists s \in \mathbb{Q} \text{ such that } s > q \text{ and } \forall p \in \alpha, p + s < 0\}.$$

We will show that $\beta$ is the additive inverse of $\alpha$. To do so we must show that $\beta$ is a cut and that $\alpha + \beta = \hat{0}$.

We would like to define the inverse $\beta$ to be

$$\{q \in \mathbb{Q} : -q \notin \alpha\}$$

but this does not work for $\hat{r}$ for $r \in \mathbb{Q}$. In such a case, $\beta$ would have a maximal element.

*First we show that $\beta$ is a cut by verifying the three cut properties.*

1. Since $\alpha \neq \mathbb{Q}$ (why?), there exists $d \in \mathbb{Q}$ such that $d \notin \alpha$. By Exercise 1.2.3 $p < d$ for all $p \in \alpha$. So $p - d < 0$ for all $p \in \alpha$. Let $s = -d$ and $q = -d - 1$. Then $s > q$ and both are rational since $d$ is. Then for any $p \in \alpha$,

$$p + s = p - d < 0.$$

   So $q \in \beta$, so $\beta \neq \emptyset$.

   To show that $\beta \neq \mathbb{Q}$, let $p \in \alpha$. Let $q = -p$. If $s$ is any rational such that $s > q$, then

$$p + s > p + q = p - p = 0.$$

   So $q \notin \beta$, so $\beta \neq \mathbb{Q}$.

2. Let $q \in \beta$ and let $a$ be a rational with $a < q$. (Show $a \in \beta$.) By definition of $\beta$, there exists a rational $s$ so that $s > q$ and $p + s < 0$ for all $p \in \alpha$. But $a < q < s$ and we still have and $p + s < 0$ for all $p \in \alpha$. So $a \in \beta$.

3. Let $q \in \beta$. (Find $r \in \mathbb{Q}$ so that $r > q$ and $r \in \beta$.) By definition of $\beta$ there is a rational $s$ so that $s > q$ and $p + s < 0$ for all $p \in \alpha$. Let $r = \frac{s+q}{2}$. Then $r$ is rational since $s$ and $q$ are. But

$$\frac{q + q}{2} < \frac{s + q}{2} < \frac{s + s}{2}.$$

   In other words, $q < r < s$. Since we know $p + s < 0$ for all $p \in \alpha$, it follows that $r \in \beta$.

So $\beta$ is a cut.

*Now we show that* $\alpha + \beta = \hat{0}$. This requires two subset proofs. First we show $\alpha + \beta \subseteq \hat{0}$. Let $r \in \alpha$ and $q \in \beta$. By Corollary 1.2.17 we must show $r + q \in \hat{0}$, that is $r + q < 0$. Since $q \in \beta$, there is a rational $s$ such that $s > q$ and $p + s < 0$ for all $p \in \alpha$. In particular, since $r \in \alpha$, we have $r + s < 0$. But then $r + q < r + s < 0$, so $r + q \in \hat{0}$.

Next we show $\hat{0} \subseteq \alpha + \beta$. *This is the hardest part of the proof.*

Let $t \in \hat{0}$. (We must show that $t = r + q$, where $r \in \alpha$ and $q \in \beta$.) By Theorem 1.2.5, $t < 0$. Now take any $a \in \alpha$ and $b \in \beta$. By definition of $\beta$ there exists $s > b$ so that $p + s < 0$ for all $p \in \alpha$. Since $a \in \alpha$ and $s > b$, it follows that $a + b < a + s < 0$.

Since $\alpha$ is a cut there exists $c \in \mathbb{Q}$ with $c \notin \alpha$. By Exercise 1.2.3 $c > a$. Choose $n \in \mathbb{N}$ so that $n > \frac{c-a}{-t}$. It follows that $a - nt > c$. Consider the finite (increasing) list of rationals:

$$a, a - \tfrac{t}{2}, a - \tfrac{2t}{2}, a - \tfrac{3t}{2}, a - \tfrac{4t}{2}, \ldots, a - \tfrac{2nt}{2} = a - nt.$$

Since $a - nt > c$ and $c \notin \alpha$, by Exercise 1.2.4 it follows that $a - nt \notin \alpha$. So in the finite list above, there is a largest element $a - \frac{kt}{2} \in \alpha$. This means that both $a - \frac{(k+1)t}{2}$ and $a - \frac{(k+2)t}{2}$ are not in $\alpha$.

Let $r = a - \frac{kt}{2}$, $q = -a + \frac{(k+2)t}{2}$, and $s = -a + \frac{(k+1)t}{2}$, all of which are rational since $a, k, t \in \mathbb{Q}$. Since $t < 0$ and $q = s + \frac{t}{2}$, it follows that $s > q$. Notice that for all $p \in \alpha$, since $a - \frac{(k+1)t}{2} \notin \alpha$ by Exercise 1.2.3 it follows that $p < a - \frac{(k+1)t}{2}$. Thus for all $p \in \alpha$,

$$s + p < \left(-a + \frac{(k+1)t}{2}\right) + \left(a - \frac{(k+1)t}{2}\right) = 0.$$

Since $s > q$, by definition $q \in \beta$.

Finally,

$$r + q = \left(a - \tfrac{kt}{2}\right) + \left(-a + \tfrac{(k+2)t}{2}\right) = \tfrac{2t}{2} = t,$$

where $r \in \alpha$ and $q \in \beta$. So $t \in \alpha + b$. $\square$

**THEOREM 1.2.25.** Additive inverses are unique. That is, if $\alpha, \beta, \lambda \in \mathbb{R}$ and $\alpha + \beta = \hat{0} = \alpha + \lambda$, then $\beta = \lambda$.

**EXERCISE 1.2.26.** Prove this result. Hint: Consider the expression $\beta + \alpha + \lambda$.

**EXERCISE 1.2.27.** Let $\alpha \in \mathbb{R}$ (cut). If $a \in \alpha$ and $b \in (-\alpha)$, then $a < -b$.

This is the same as saying: If $a \in \alpha$ and $a \geq -b$, then $-b \notin (-\alpha)$.

This is the same as saying: If $b \in (-\alpha)$ and $a \geq -b$, then $a \notin \alpha$.